电动力学

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目录

1　数学基础

1.1　向量代数

1. 运算规则：
   (1) Einstein 求和约定：$\boldsymbol{A} = A_i\boldsymbol{e}_i$
   (2) Kronecher δ 符号：$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
   (3) Levi-Civita 符号：$\varepsilon_{ijk} = \begin{cases} 1 & i,j,k \text{ 偶排列} \\ -1 & i,j,k \text{ 奇排列} \\ 0 & i,j,k \text{ 有相同者} \end{cases}$
   (4) 单位全反对称张量乘积公式：$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$
2. 向量代数：
   (1) 加法：$\boldsymbol{A} + \boldsymbol{B} = (A_1 + B_1)\boldsymbol{e}_1 + (A_2 + B_2)\boldsymbol{e}_2 + (A_3 + B_3)\boldsymbol{e}_3 = (A_i + B_i)\boldsymbol{e}_i$
   (2) 数乘：$\alpha\boldsymbol{A} = \alpha A_1\boldsymbol{e}_1 + \alpha A_2\boldsymbol{e}_2 + \alpha A_3\boldsymbol{e}_3 = \alpha A_i\boldsymbol{e}_i$
   (3) 标积：$\boldsymbol{A} \cdot \boldsymbol{B} = A_1B_1 + A_2B_2 + A_3B_3 = A_iB_i$
   (4) 矢积：$\boldsymbol{A} \times \boldsymbol{B} = \varepsilon_{ijk}\boldsymbol{e}_iA_jB_k$
3. 重要结论：
   (1) $\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A})$ (1.1)
   (2) $\boldsymbol{A} \times \boldsymbol{B} \times \boldsymbol{C} = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ (1.2)
   (3) $(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C})$ (1.3)

1.2　向量分析

1. Nabla 算符：

$$\nabla = \partial_i\boldsymbol{e}_i \tag{1.4}$$

2. Laplace 算符：

$$\nabla^2 = \nabla \cdot \nabla = (\boldsymbol{e}_i\partial_i) \cdot (\boldsymbol{e}_j\partial_j) = \delta_{ij}\partial_i\partial_j = \partial_i\partial_i \tag{1.5}$$

3. 标量场的梯度：

$$\mathrm{grad}\ \varphi = \nabla\varphi = \boldsymbol{e}_i\partial_i\varphi \tag{1.6}$$

4. 向量场的散度：

$$\mathrm{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \partial_iA_i \tag{1.7}$$

5. 向量场的旋度：

$$\mathrm{rot}\boldsymbol{A} = \nabla \times \boldsymbol{A} = \varepsilon_{ijk}\boldsymbol{e}_i\partial_jA_k \tag{1.8}$$

6. Gauss 公式：

$$\oint_{\partial V} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{A}\mathrm{d}V \tag{1.9}$$

7. Stokes 公式：

$$\oint_{\partial S} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\nabla \times \boldsymbol{A}) \cdot \mathrm{d}\boldsymbol{S} \tag{1.10}$$

8. Green 公式：

$$\int_{\partial V} (\psi\nabla\varphi - \varphi\nabla\psi) \cdot \mathrm{d}\boldsymbol{S} = \int_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\mathrm{d}V \tag{1.11}$$

1. $\nabla r = \frac{\boldsymbol{r}}{r}$ (1.12)
2. $\nabla \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3}$ (1.13)
3. $\nabla \cdot \boldsymbol{r} = 3$ (1.14)
4. $\nabla \times \boldsymbol{r} = 0$ (1.15)
5. $\nabla \cdot \frac{\boldsymbol{r}}{r^3} = 0$ (1.16)
6. $\nabla \times \frac{\boldsymbol{r}}{r^3} = 0$ (1.17)
7. $\nabla \times \nabla \varphi = 0$ (1.18)
8. $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ (1.19)
9. $\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B})$ (1.20)
10. $\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{A}(\nabla \cdot \boldsymbol{B})$ (1.21)
11. $\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} + (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + \boldsymbol{B} \times \nabla \times \boldsymbol{A} + \boldsymbol{A} \times \nabla \times \boldsymbol{B}$ (1.22)
12. $\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ (1.23)
13. $(\nabla \times \boldsymbol{A}) \times \boldsymbol{A} = (\boldsymbol{A} \cdot \nabla)\boldsymbol{A} - \frac{1}{2}\nabla \boldsymbol{A}^2$ (1.24)

1.3 张量分析

1. 并矢：将两个矢量并列，不做任何标积和矢积的运算。是一种特殊的二阶张量。

$$\boldsymbol{AB} = A_i B_j \boldsymbol{e}_i \boldsymbol{e}_j \tag{1.25}$$

2. 张量：张量是一个多维数组，用于表示在多个维度上变化的量。
   (1) 0 阶张量：标量
   (2) 1 阶张量：矢量，$\boldsymbol{A} = A_i \boldsymbol{e}_i$
   (3) 2 阶张量：矩阵，$\overleftrightarrow{\boldsymbol{T}} = T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$
3. 并矢代数：
   (1) $(\boldsymbol{AB}) \cdot \boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})$, $\boldsymbol{C} \cdot (\boldsymbol{AB}) = (\boldsymbol{C} \cdot \boldsymbol{A})\boldsymbol{B}$ (1.26)
   (2) $(\boldsymbol{AB}) \cdot (\boldsymbol{CD}) = \boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C})\boldsymbol{D}$ (1.27)
   (3) $(\boldsymbol{AB}) : (\boldsymbol{CD}) = (\boldsymbol{B} \cdot \boldsymbol{C})(\boldsymbol{A} \cdot \boldsymbol{D})$ (1.28)
4. 张量代数：
   (1) $\boldsymbol{A} \cdot \overleftrightarrow{\boldsymbol{T}} = (A_i \boldsymbol{e}_i) \cdot (T_{jk} \boldsymbol{e}_j \boldsymbol{e}_k) = A_i T_{ik} \boldsymbol{e}_k = A_i T_{ij} \boldsymbol{e}_j$ (1.29)
   (2) $\overleftrightarrow{\boldsymbol{T}} \cdot \boldsymbol{A} = (T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j) \cdot (A_k \boldsymbol{e}_k) = T_{ik} A_k \boldsymbol{e}_i = T_{ji} A_i \boldsymbol{e}_j$ (1.30)
   (3) $\overleftrightarrow{\boldsymbol{T}} \cdot \overleftrightarrow{\boldsymbol{S}} = (T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j) \cdot (S_{kl} \boldsymbol{e}_k \boldsymbol{e}_l) = T_{ij} S_{jl} \boldsymbol{e}_i \boldsymbol{e}_l = T_{ij} S_{jk} \boldsymbol{e}_i \boldsymbol{e}_k$ (1.31)
   (4) $\overleftrightarrow{\boldsymbol{T}} : \overleftrightarrow{\boldsymbol{S}} = (T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j) : (S_{kl} \boldsymbol{e}_k \boldsymbol{e}_l) = T_{ij} S_{ji}$ (1.32)
   (5) $\overleftrightarrow{\boldsymbol{I}} = \delta_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$ (1.33)
5. 张量分析：
   (1) $\nabla \boldsymbol{r} = (\partial_i \boldsymbol{e}_i)(x_j \boldsymbol{e}_j) = \delta_{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \overleftrightarrow{\boldsymbol{I}}$ (1.34)
   (2) $\nabla \boldsymbol{A} = (\partial_i \boldsymbol{e}_i)(A_j \boldsymbol{e}_j) = \partial_i A_j \boldsymbol{e}_i \boldsymbol{e}_j$ (矢量的梯度是张量) (1.35)
   (3) $\nabla \cdot \overleftrightarrow{\boldsymbol{T}} = \partial_i (T_{ij} \boldsymbol{e}_i \boldsymbol{e}_j)_i = \partial_i T_{ij} \boldsymbol{e}_j$ (张量的散度是矢量) (1.36)

1.4 曲线系向量分析

1. 一般正交曲线系：

(1) 梯度：

$$\nabla = \frac{1}{h_1}\frac{\partial}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial}{\partial u_3}\boldsymbol{e}_3 \tag{1.37}$$

(2) 散度：

$$\nabla\cdot\boldsymbol{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right] \tag{1.38}$$

(3) 旋度：

$$\begin{aligned}\nabla\times\boldsymbol{A} =& \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(A_3h_3) - \frac{\partial}{\partial u_3}(A_2h_2)\right]\boldsymbol{e}_1 + \\ & \frac{1}{h_3h_1}\left[\frac{\partial}{\partial u_3}(A_1h_1) - \frac{\partial}{\partial u_1}(A_3h_3)\right]\boldsymbol{e}_2 + \\ & \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(A_2h_2) - \frac{\partial}{\partial u_2}(A_1h_1)\right]\boldsymbol{e}_3\end{aligned} \tag{1.39}$$

(4) Laplace 算符：

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right] \tag{1.40}$$

2. 柱坐标 (ρ,φ,z)：$h_\rho = 1,\ h_\varphi = \rho,\ h_z = 1$

(1) 梯度：

$$\nabla = \frac{\partial}{\partial\rho}\boldsymbol{e}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi + \frac{\partial}{\partial z}\boldsymbol{e}_z \tag{1.41}$$

(2) 散度：

$$\nabla\cdot\boldsymbol{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(A_\rho\rho) + \frac{\partial}{\partial\varphi}A_\varphi + \frac{\partial}{\partial z}(A_z\rho)\right] \tag{1.42}$$

(3) 旋度：

$$\nabla\times\boldsymbol{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial\varphi}A_z - \frac{\partial}{\partial z}(\rho A_\varphi)\right]\boldsymbol{e}_\rho + \left[\frac{\partial}{\partial z}A_\rho - \frac{\partial}{\partial\rho}A_z\right]\boldsymbol{e}_\varphi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\varphi) - \frac{\partial}{\partial\varphi}A_\rho\right]\boldsymbol{e}_z \tag{1.43}$$

(4) Laplace 算符：

$$\nabla^2 = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial}{\partial z}\right)\right] \tag{1.44}$$

3. 球坐标 (r,θ,φ)：$h_r = 1,\ h_\theta = r,\ h_\varphi = r\sin\theta$

(1) 梯度：

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi \tag{1.45}$$

(2) 散度：

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(A_r r^2\sin\theta) + \frac{\partial}{\partial\theta}(A_\theta r\sin\theta) + \frac{\partial}{\partial\varphi}(A_\varphi r)\right] \tag{1.46}$$

(3) 旋度：

$$\begin{aligned}\nabla\times\boldsymbol{A} =& \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta A_\varphi) - \frac{\partial}{\partial\varphi}(rA_\theta)\right]\boldsymbol{e}_r + \\ & \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\varphi}A_r - \frac{\partial}{\partial r}(r\sin\theta A_\varphi)\right]\boldsymbol{e}_\theta + \\ & \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\boldsymbol{e}_\varphi\end{aligned} \tag{1.47}$$

(4) Laplace 算符：

$$\nabla^2 = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\right] \tag{1.48}$$

1.5 δ 函数

1. δ 函数：满足如下两个条件的泛函。

(1) $\delta(x-x_0)=\begin{cases}+\infty, & x-x_0=0\\ 0, & x-x_0\neq 0\end{cases}$ (1.49)

(2) $\int_{-\infty}^{+\infty}\delta(x-x_0)\mathrm{d}x=1$ (1.50)

2. δ 函数的三维形式：

(1) 直角坐标：

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{1.51}$$

(2) 柱坐标 (ρ,θ,z)：

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=\frac{1}{\rho}\delta(\rho-\rho_0)\delta(\theta-\theta_0)\delta(z-z_0) \tag{1.52}$$

(3) 球坐标 (r,θ,φ)：

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=\frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0) \tag{1.53}$$

(4) Possion 形式：

$$\delta(\boldsymbol{r}-\boldsymbol{r}_0)=-\frac{1}{4\pi}\nabla^2\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{1.54}$$

3. δ 函数的其他形式：

(1) $\delta(x)=H'(x)$ (1.55)

(2) $\delta(x)=\lim\limits_{\alpha\to 0}\delta(x,\alpha)=\lim\limits_{\alpha\to 0}\frac{1}{\pi}\int_0^{+\infty}\cos kx\mathrm{e}^{-\alpha k}\mathrm{d}k=\lim\limits_{\alpha\to 0}\frac{1}{\pi}\frac{\alpha}{\alpha^2+x^2}$ (1.56)

(3) $\delta(x)=\lim\limits_{n\to+\infty}\sqrt{\frac{n}{\pi}}\mathrm{e}^{nx^2}$ (1.57)

(4) $\delta(x)=\lim\limits_{n\to+\infty}\frac{1}{\pi}\int_0^n\cos kx\mathrm{d}x=\lim\limits_{n\to+\infty}\frac{\sin nx}{\pi x}$ (1.58)

4. δ 函数的性质：

(1) $\delta(-x)=\delta(x)$ (1.59)

(2) $f(x)\delta(x-x_0)=f(x_0)\delta(x-x_0)$ (1.60)

(3) $\int_{-\infty}^{+\infty}f(x)\delta(x-x_0)\mathrm{d}x=f(x_0)$ (1.61)

(4) $\delta(\varphi(x))=\sum\limits_{i=1}^{k}\frac{1}{|\varphi'(x_i)|}\delta(x-x_i)$ (1.62)

$\Rightarrow \delta(ax)=\frac{1}{|a|}\delta(x),\ \delta(x^2-a^2)=\frac{1}{2|a|}(\delta(x+a)+\delta(x-a))$ (1.63)

5. δ 函数导数的性质：

(1) $\delta^{(n)}(-x)=(-1)^n\delta(x)$ (1.64)

(2) $\int_{-\infty}^{+\infty}f(x)\delta^{(n)}(x-x_0)\mathrm{d}x=(-1)^n f^{(n)}(x_0)$ (1.65)

2 真空中的麦克斯韦方程组

2.1 电荷和电场

2.1.1 Coulomb 定律

1. Coulomb 定律:

$$\boldsymbol{F}_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \tag{2.1}$$

2. Coulomb 定律遵循叠加原理:

$$\boldsymbol{F} = \sum_i F_i \tag{2.2}$$

3. 点电荷的电场:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}q} = \frac{q}{4\pi\varepsilon_0} \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \tag{2.3}$$

4. 连续带电体的电场:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \rho(\boldsymbol{r}) \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}V' \tag{2.4}$$

2.1.2 电场的散度

1. 静电场 Gauss 定理:

$$\oint_S \boldsymbol{E}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = \frac{Q_{in}}{\varepsilon_0} \tag{2.5}$$

2. 静电场的散度:

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r})}{\varepsilon_0} \tag{2.6}$$

Proof. 考虑一个点电荷 q 以及一个闭合曲面，选取面元 $\mathrm{d}\boldsymbol{S}$，那么

$$\boldsymbol{E}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = E(\boldsymbol{r}) \cos\theta \mathrm{d}S = \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \cos\theta \mathrm{d}S$$

使用立体角，我们有

$$\cos\theta \mathrm{d}S = r^2 \mathrm{d}\Omega$$

因此

$$\oint_S \boldsymbol{E}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = \frac{q}{4\pi\varepsilon_0} \int \mathrm{d}\Omega$$

当点电荷位于闭合曲面中时，立体角积分为 4π，否则为 0。对于分立电荷和连续电荷可以直接推广。 □

2.1.3 电场的旋度

1. 静电场环路定理:

$$\oint_L \boldsymbol{E}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = 0 \tag{2.7}$$

2. 静电场的旋度:

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \tag{2.8}$$

Proof. 考虑单个点电荷，

$$\oint_L \boldsymbol{E}(\boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\varepsilon_0} \oint_L \frac{\boldsymbol{r}}{r^3} \cdot \mathrm{d}\boldsymbol{l} = \frac{q}{4\pi\varepsilon_0} \oint_L \frac{1}{r^3} \cdot r\mathrm{d}r = -\frac{q}{4\pi\varepsilon_0} \oint_L \mathrm{d}\left(\frac{1}{r}\right) = 0$$

对于分立电荷和连续电荷可以直接推广。 □

1. 点电荷的电势：

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{2.9}$$

2. 连续带电体的电势：

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' \tag{2.10}$$

3. 电场是电势的负梯度：

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\varphi(\boldsymbol{r}) \tag{2.11}$$

4. Possion 方程：

$$\nabla^2\varphi(\boldsymbol{r}) = -\frac{\rho(\boldsymbol{r})}{\varepsilon_0} \tag{2.12}$$

5. Laplace 方程：

$$\nabla^2\varphi(\boldsymbol{r}) = 0 \tag{2.13}$$

2.2 电流和磁场

2.2.1 电流密度和电荷守恒定律

1. 电荷密度：

$$\rho(\boldsymbol{r},t) = \lim_{\Delta V\to 0}\frac{\Delta Q}{\Delta V} \Rightarrow Q(t) = \int_V \rho(\boldsymbol{r},t)\mathrm{d}V \tag{2.14}$$

2. 电流密度：

$$\boldsymbol{j}(\boldsymbol{r},t) = \rho(\boldsymbol{r})\boldsymbol{v} \tag{2.15}$$

3. 电流强度：

$$I(t) = \int_S \boldsymbol{j}(\boldsymbol{r},t)\mathrm{d}\boldsymbol{S} \tag{2.16}$$

4. 电荷守恒定律：

$$\frac{\partial\rho(\boldsymbol{r},t)}{\partial t} + \nabla\cdot\boldsymbol{j}(\boldsymbol{r},t) = 0 \tag{2.17}$$

2.2.2 Biot-Savart 定律

1. 电流元 $I\mathrm{d}\boldsymbol{l}$ 在磁场中的受力：

$$\mathrm{d}\boldsymbol{F} = I\mathrm{d}\boldsymbol{l}\times\boldsymbol{B} \tag{2.18}$$

2. Biot-Savart 定律：

$$\mathrm{d}B(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\boldsymbol{l}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{2.19}$$

$$B(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{V'}\boldsymbol{j}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V' \tag{2.20}$$

3. 闭合细导线产生的磁感应强度：$\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V' = I\mathrm{d}\boldsymbol{l}$

$$B(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\oint_L\frac{I\mathrm{d}\boldsymbol{l}\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{2.21}$$

2.2.3 磁场的散度

1. 静磁场 Gauss 定理：

$$\oint_S B(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S} = 0 \tag{2.22}$$

2. 静磁场的旋度：

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r}) = 0 \tag{2.23}$$

Proof. 从 Biot-Savart 定律出发，有

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0}{4\pi}\int_{V'}\boldsymbol{j}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V' \quad \text{（对 } r \text{ 作梯度，利用式(1.13)）}\\ &= -\frac{\mu_0}{4\pi}\int_{V'}\boldsymbol{j}(\boldsymbol{r}')\times\nabla_r\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' = \frac{\mu_0}{4\pi}\int_{V'}\nabla_r\times\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' = \nabla_r\times\boldsymbol{A}(\boldsymbol{r})\end{aligned}$$

其中，磁矢势 $\boldsymbol{A}(\boldsymbol{r})$ 定义为

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{V'}\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'$$

因此，再利用式(1.19)可得

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r}) = \nabla\cdot(\nabla\times\boldsymbol{A}(\boldsymbol{r})) = 0$$ □

2.2.4 磁场的旋度

1. Ampere 环路定理：

$$\oint_L \boldsymbol{B}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l} = \mu_0 I = \mu_0\int_S \boldsymbol{j}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S} \tag{2.24}$$

2. 静磁场的旋度：

$$\nabla\times\boldsymbol{B}(\boldsymbol{r}) = \mu_0\boldsymbol{j}(\boldsymbol{r}) \tag{2.25}$$

Proof. 从矢量势出发计算静磁场的旋度：

$$\begin{aligned}\nabla\times\boldsymbol{B}(\boldsymbol{r}) &= \nabla\times(\nabla\times\boldsymbol{A}(\boldsymbol{r})) = \nabla(\nabla\cdot\boldsymbol{A}(\boldsymbol{r})) - \nabla^2\boldsymbol{A}(\boldsymbol{r})\\ &= \frac{\mu_0}{4\pi}\nabla_r\left[\int_{V'}\nabla_r\cdot\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right] - \frac{\mu_0}{4\pi}\int_{V'}\nabla_r^2\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\end{aligned}$$

先计算第一项。因为积分区域包括了所有电流，没有电流通过界面，

$$\int_{V'}\nabla_{r'}\cdot\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' = \oint_S\frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\cdot\mathrm{d}\boldsymbol{S} = 0$$

因此第一项为零，只需要计算第二项

$$\nabla\times\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi}\int_{V'}\boldsymbol{j}(\boldsymbol{r}')\nabla_r^2\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' = \frac{\mu_0}{4\pi}\int_{V'}\boldsymbol{j}(\boldsymbol{r}')4\pi\delta(\boldsymbol{r}-\boldsymbol{r}')\mathrm{d}V' = \mu_0\boldsymbol{j}(\boldsymbol{r})$$ □

2.3 含时电磁场

2.3.1 Faraday 电磁感应定律

1. Faraday 电磁感应定律：

$$\oint_L \boldsymbol{E}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} \tag{2.26}$$

2. 电场的旋度：

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \tag{2.27}$$

Proof. Neumann 首次给出的数学形式为

$$\varepsilon(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S}$$

感应电动势又可以写成电场强度沿闭合回路的线积分

$$\varepsilon(t) = \oint_L \boldsymbol{E}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{l} = \int_S(\nabla\times\boldsymbol{E}(\boldsymbol{r},t))\cdot\mathrm{d}\boldsymbol{S}$$

联立以上两式可得 Faraday 电磁感应定律。 □

2.3.2 位移电流

1. 电荷守恒定律给出：

$$\nabla \cdot \boldsymbol{j}(\boldsymbol{r}, t) = -\frac{\partial \rho(\boldsymbol{r}, t)}{\partial t} \tag{2.28}$$

   磁场的旋度式(2.25)取散度：

$$\nabla \cdot \boldsymbol{j}(\boldsymbol{r}, t) = 0 \tag{2.29}$$

   发现它们并不自洽。
2. 电场的散度式(2.6)取微分：

$$\varepsilon_0 \nabla \cdot \frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} = \frac{\partial \rho(\boldsymbol{r}, t)}{\partial t} \tag{2.30}$$

3. Maxwell 将磁场的旋度修改为：

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}, t) = \mu_0 \boldsymbol{j}(\boldsymbol{r}, t) + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} \tag{2.31}$$

4. 位移电流密度：

$$\boldsymbol{j}_D(\boldsymbol{r}, t) = \varepsilon_0 \frac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} \tag{2.32}$$

2.3.3 真空中的 Maxwell 方程组

$$\begin{cases} \nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = \dfrac{\rho(\boldsymbol{r}, t)}{\varepsilon_0} \\ \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\dfrac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} \\ \nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0 \\ \nabla \times \boldsymbol{B}(\boldsymbol{r}, t) = \mu_0 \boldsymbol{j}(\boldsymbol{r}, t) + \mu_0 \varepsilon_0 \dfrac{\partial \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t} \end{cases} \tag{2.33}$$

2.3.4 Lorentz 力公式

1. 点电荷在电磁场中的受力：

$$\boldsymbol{F}(\boldsymbol{r}, t) = q\boldsymbol{E}(\boldsymbol{r}, t) + q\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t) \tag{2.34}$$

2. 连续带电体在电磁场中的受力密度：

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{r}, t) &= \rho(\boldsymbol{r}, t)\boldsymbol{E}(\boldsymbol{r}, t) + \rho(\boldsymbol{r}, t)\boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t) \\ &= \rho(\boldsymbol{r}, t)\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{j}(\boldsymbol{r}, t) \times \boldsymbol{B}(\boldsymbol{r}, t) \end{aligned} \tag{2.35}$$

2.4 电磁规律的守恒律

2.4.1 电荷守恒

1. 电荷守恒定律：

$$\frac{\partial \rho(\boldsymbol{r}, t)}{\partial t} + \nabla \cdot \boldsymbol{j}(\boldsymbol{r}, t) = 0 \tag{2.36}$$

Proof. 由 Maxwell 方程组(2.33)式四取散度可得

$$\nabla \cdot (\nabla \times \boldsymbol{B}(\boldsymbol{r}, t)) = \mu_0 \nabla \cdot \boldsymbol{j}(\boldsymbol{r}, t) + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t)) = 0$$

式一对时间求偏导可得

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t)) = \frac{1}{\varepsilon_0} \frac{\partial \rho(\boldsymbol{r}, t)}{\partial t}$$

联立以上两式可得电荷守恒定律。 □

2.4.2 能量守恒

1. Poynting 定理:

$$\frac{\partial\omega(\boldsymbol{r},t)}{\partial t}+\nabla\cdot\boldsymbol{S}(\boldsymbol{r},t)+\boldsymbol{j}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)=0 \tag{2.37}$$

2. 能量密度:

$$w(\boldsymbol{r},t)=\frac{1}{2}\left(\varepsilon_0E^2(\boldsymbol{r},t)+\frac{1}{\mu_0}B^2(\boldsymbol{r},t)\right) \tag{2.38}$$

3. 能流密度:

$$\boldsymbol{S}(\boldsymbol{r},t)=\frac{1}{\mu_0}(\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)) \tag{2.39}$$

Proof. 区域 V 内的能量守恒方程应该写为

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V w(\boldsymbol{r},t)\mathrm{d}V=-\oint_S\boldsymbol{S}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S}-\int_V\frac{\partial}{\partial t}w_{int}(\boldsymbol{r},t)\mathrm{d}V$$

其中，$w(\boldsymbol{r},t)$ 为能量密度，$\boldsymbol{S}(\boldsymbol{r},t)$ 为能流密度，最后一项 $w_{int}(\boldsymbol{r},t)$ 表示区域内场与粒子相互作用的能量交换，具体形式可以通过 Lorentz 力写出来

$$\frac{\partial}{\partial t}w_{int}(\boldsymbol{r},t)=\boldsymbol{f}(\boldsymbol{r},t)\cdot\boldsymbol{v}=\rho(\boldsymbol{r},t)\boldsymbol{v}\cdot\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{j}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)$$

用 $\boldsymbol{E}(\boldsymbol{r},t)$ 点乘方程组(2.33)中的式四，可得

$$\boldsymbol{E}(\boldsymbol{r},t)\cdot(\nabla\times\boldsymbol{B}(\boldsymbol{r},t))=\mu_0\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{j}(\boldsymbol{r},t)+\mu_0\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}$$

再用 $\boldsymbol{B}(\boldsymbol{r},t)$ 点乘方程组(2.33)中的式二，可得

$$\boldsymbol{B}(\boldsymbol{r},t)\cdot(\nabla\times\boldsymbol{E}(\boldsymbol{r},t))=-\boldsymbol{B}(\boldsymbol{r},t)\cdot\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

上面两式相减并利用式(1.20)给出

$$\begin{aligned}\nabla\cdot(\boldsymbol{B}(\boldsymbol{r},t)\times\boldsymbol{E}(\boldsymbol{r},t))&=\boldsymbol{E}(\boldsymbol{r},t)\cdot(\nabla\times\boldsymbol{B}(\boldsymbol{r},t))-\boldsymbol{B}(\boldsymbol{r},t)\cdot(\nabla\times\boldsymbol{E}(\boldsymbol{r},t))\\&=\mu_0\boldsymbol{E}(\boldsymbol{r},t)\cdot\boldsymbol{j}(\boldsymbol{r},t)+\mu_0\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t}+\boldsymbol{B}(\boldsymbol{r},t)\cdot\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t}\\&=\mu_0\boldsymbol{j}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)+\frac{1}{2}\mu_0\varepsilon_0\frac{\partial E^2(\boldsymbol{r},t)}{\partial t}+\frac{1}{2}\frac{\partial B^2(\boldsymbol{r},t)}{\partial t}\end{aligned}$$

若令

$$\boldsymbol{S}(\boldsymbol{r},t)=\frac{1}{\mu_0}(\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)),\quad w(\boldsymbol{r},t)=\frac{1}{2}\left(\varepsilon_0E^2(\boldsymbol{r},t)+\frac{1}{\mu_0}B^2(\boldsymbol{r},t)\right)$$

此式又可以写为

$$\frac{\partial w(\boldsymbol{r},t)}{\partial t}+\nabla\cdot\boldsymbol{S}(\boldsymbol{r},t)+\boldsymbol{j}(\boldsymbol{r},t)\cdot\boldsymbol{E}(\boldsymbol{r},t)=0$$

□

2.4.3 动量守恒

1. 动量守恒:

$$\boldsymbol{f}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{g}(\boldsymbol{r},t)}{\partial t}+\nabla\cdot\overleftrightarrow{\boldsymbol{T}}(\boldsymbol{r},t)=0 \tag{2.40}$$

2. 动量密度:

$$\boldsymbol{g}(\boldsymbol{r},t)=\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)=\varepsilon_0\mu_0\boldsymbol{S}(\boldsymbol{r},t) \tag{2.41}$$

3. 动量流密度:

$$\overleftrightarrow{\boldsymbol{T}}(\boldsymbol{r},t)=\omega(\boldsymbol{r},t)\overleftrightarrow{\boldsymbol{I}}-\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t)\boldsymbol{E}(\boldsymbol{r},t)-\frac{1}{\mu_0}\boldsymbol{B}(\boldsymbol{r},t)\boldsymbol{B}(\boldsymbol{r},t) \tag{2.42}$$

Proof. 区域 V 内的动量守恒方程应该写为

$$\int_V \boldsymbol{f}(\boldsymbol{r},t)\mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t}\int_V \boldsymbol{g}(\boldsymbol{r},t)\mathrm{d}V - \oint_S \overleftrightarrow{\boldsymbol{T}}(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}$$

其中，$\boldsymbol{g}(\boldsymbol{r},t)$ 为动量密度，$\overleftrightarrow{\boldsymbol{T}}(\boldsymbol{r},t)$ 为动量流密度，这是因为因为动量密度的每个分量都有一个三维的流密度矢量，将其放在一起就构成一个张量。

利用方程组中含源的两个方程，则洛伦兹力密度 $\boldsymbol{f}(\boldsymbol{r},t)$ 可以写为

$$\boldsymbol{f} = \rho\boldsymbol{E} + \boldsymbol{j}\times\boldsymbol{B} = \varepsilon_0(\nabla\cdot\boldsymbol{E})\boldsymbol{E} + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B}$$

由另外两个方程我们凑出下式:

$$0 = \frac{1}{\mu_0}(\nabla\cdot\boldsymbol{B})\boldsymbol{B} + \varepsilon_0\left((\nabla\times\boldsymbol{E}) + \frac{\partial\boldsymbol{B}}{\partial t}\right)\times\boldsymbol{E}$$

上面两式相加，可得一个关于 $\boldsymbol{E}(\boldsymbol{r},t)$ 和 $\boldsymbol{B}(\boldsymbol{r},t)$ 对称的形式:

$$\boldsymbol{f} = \varepsilon_0\left[(\nabla\cdot\boldsymbol{E})\boldsymbol{E} + (\nabla\times\boldsymbol{E})\times\boldsymbol{E}\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\boldsymbol{B})\boldsymbol{B} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}\right] - \varepsilon_0\frac{\partial}{\partial t}\left[\boldsymbol{E}\times\boldsymbol{B}\right]$$

通过张量分析我们有

$$\nabla\cdot(\boldsymbol{A}\boldsymbol{A}) = (\nabla\cdot\boldsymbol{A})\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{A},\quad (\nabla\times\boldsymbol{A})\times\boldsymbol{A} = (\boldsymbol{A}\cdot\nabla)\boldsymbol{A} - \frac{1}{2}\nabla\boldsymbol{A}^2,$$

利用这两个公式我们可以得到

$$(\nabla\cdot\boldsymbol{E})\boldsymbol{E} + (\nabla\times\boldsymbol{E})\times\boldsymbol{E} = \nabla\cdot\left[\boldsymbol{E}\boldsymbol{E} - \frac{1}{2}E^2\overleftrightarrow{\boldsymbol{I}}\right]$$

$$(\nabla\cdot\boldsymbol{B})\boldsymbol{B} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \nabla\cdot\left[\boldsymbol{B}\boldsymbol{B} - \frac{1}{2}B^2\overleftrightarrow{\boldsymbol{I}}\right]$$

若令

$$\boldsymbol{g} = \varepsilon_0\boldsymbol{E}\times\boldsymbol{B} = \varepsilon_0\mu_0\boldsymbol{S},\quad \overleftrightarrow{\boldsymbol{T}} = \frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)\overleftrightarrow{\boldsymbol{I}} - \varepsilon_0\boldsymbol{E}\boldsymbol{E} - \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{B}$$

此式又可以写为

$$\boldsymbol{f}(\boldsymbol{r},t) + \frac{\partial\boldsymbol{g}(\boldsymbol{r},t)}{\partial t} + \nabla\cdot\overleftrightarrow{\boldsymbol{T}}(\boldsymbol{r},t) = 0$$

□

2.4.4 角动量守恒

1. 场和源总角动量守恒:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V(\boldsymbol{J}_{src} + \boldsymbol{J}_{field})\mathrm{d}V = -\oint_S \overleftrightarrow{\boldsymbol{M}}\cdot\mathrm{d}\boldsymbol{S} \tag{2.43}$$

2. 场的角动量密度:

$$\boldsymbol{J}_{field}(\boldsymbol{r},t) = \boldsymbol{r}\times\boldsymbol{g}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{r}\times(\boldsymbol{E}(\boldsymbol{r},t)\times\boldsymbol{B}(\boldsymbol{r},t)) \tag{2.44}$$

3. 源的角动量密度:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\boldsymbol{J}_{src}(\boldsymbol{r},t)\mathrm{d}V = \int(\boldsymbol{r}\times\boldsymbol{f}(\boldsymbol{r},t))\mathrm{d}V \tag{2.45}$$

4. 角动量流密度:

$$\overleftrightarrow{\boldsymbol{M}}(\boldsymbol{r},t) = \boldsymbol{r}\times\overleftrightarrow{\boldsymbol{T}}(\boldsymbol{r},t) \tag{2.46}$$

2.5 电磁势与规范对称性

1. 电磁场的矢势：

$$\boldsymbol{B}(\boldsymbol{r},t)=\nabla\times\boldsymbol{A}(\boldsymbol{r},t) \tag{2.47}$$

2. Maxwell 方程组第二个方程改写为：

$$\nabla\times\left(\boldsymbol{E}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}\right)=0 \tag{2.48}$$

3. 电磁场的标势：

$$\boldsymbol{E}(\boldsymbol{r},t)+\frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}=-\nabla\varphi(\boldsymbol{r},t) \tag{2.49}$$

4. 电磁势的规范变换：通过标量场 $\varLambda(\boldsymbol{r},t)$ 连接

$$\boldsymbol{A}'(\boldsymbol{r},t)=\boldsymbol{A}(\boldsymbol{r},t)+\nabla\varLambda(\boldsymbol{r},t),\quad \varphi'(\boldsymbol{r},t)=\varphi(\boldsymbol{r},t)-\frac{\partial\varLambda(\boldsymbol{r},t)}{\partial t} \tag{2.50}$$

5. 电磁势的规范对称性：

$$\boldsymbol{B}'(\boldsymbol{r},t)=\nabla\times\boldsymbol{A}'(\boldsymbol{r},t)=\nabla\times\boldsymbol{A}(\boldsymbol{r},t)+\nabla\times(\nabla\varLambda(\boldsymbol{r},t))=\boldsymbol{B}(\boldsymbol{r},t) \tag{2.51}$$

$$\boldsymbol{E}'(\boldsymbol{r},t)=-\nabla\varphi'(\boldsymbol{r},t)-\frac{\partial\boldsymbol{A}'(\boldsymbol{r},t)}{\partial t}=-\nabla\varphi(\boldsymbol{r},t)-\frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}=\boldsymbol{E}(\boldsymbol{r},t) \tag{2.52}$$

6. 规范条件：

(1) Coulomb 规范（辐射规范）：

$$\nabla^2\varLambda(\boldsymbol{r},t)=-\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t) \tag{2.53}$$

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)=0 \tag{2.54}$$

(2) Lorentz 规范（协变规范）：

$$\nabla^2\varLambda(\boldsymbol{r},t)-\mu_0\varepsilon_0\frac{\partial^2\varLambda(\boldsymbol{r},t)}{\partial t^2}=-\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)-\mu_0\varepsilon_0\frac{\partial^2\varphi(\boldsymbol{r},t)}{\partial^2 t} \tag{2.55}$$

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)+\frac{1}{c^2}\frac{\partial\varphi(\boldsymbol{r},t)}{\partial t}=0 \tag{2.56}$$

2.6 真空中 Maxwell 方程组的对称性

2.6.1 线性

2.6.2 连续对称性：规范对称性、Lorentz 协变性

1. 规范对称性：Maxwell 方程组在规范变换下具有不变性。
2. Lorentz 协变性：Maxwell 方程组在 Lorentz 变换下具有不变性，与狭义相对论完全兼容。
3. 根据 Noether 定理，Lorentz 协变性对应着能动量守恒，规范对称对应着电荷守恒。

2.6.3 分立对称性：空间反演对称性、时间反演对称性

1. 宇称算符 $\hat{P}$：

$$\hat{P}\boldsymbol{r}=-\boldsymbol{r},\quad \hat{P}\nabla=-\nabla,\quad \hat{P}\frac{\partial}{\partial t}=\frac{\partial}{\partial t},\quad \hat{P}\rho=\rho,\quad \hat{P}\boldsymbol{j}=-\boldsymbol{j} \tag{2.57}$$

2. Maxwell 方程空间反演不变：

$$\hat{P}\boldsymbol{E}=-\boldsymbol{E},\quad \hat{P}\boldsymbol{B}=\boldsymbol{B} \tag{2.58}$$

3. 时间反演算符 $\hat{K}$：

$$\hat{K}t=-t,\quad \hat{K}\nabla=\nabla,\quad \hat{K}\frac{\partial}{\partial t}=-\frac{\partial}{\partial t},\quad \hat{K}\rho=\rho,\quad \hat{K}\boldsymbol{j}=-\boldsymbol{j} \tag{2.59}$$

4. Maxwell 方程时间反演不变：

$$\hat{K}\boldsymbol{E}=\boldsymbol{E},\quad \hat{K}\boldsymbol{B}=-\boldsymbol{B} \tag{2.60}$$

3 狭义相对论电动力学

3.1 狭义相对论基本原理

3.1.1 狭义相对论时空观

1. 狭义相对论基本假设：
   (1) 相对性原理：任何惯性系中物理规律相同。
   (2) 光速不变原理：信号最大传播速度是真空中的光速。
2. 两事件的不变间隔 Δs：
$$\Delta s^2 = c^2\Delta t^2 - \Delta \boldsymbol{x}^2 = 0 \tag{3.1}$$
   有三种分类：$\Delta s^2 > 0$ 类时间隔，$\Delta s^2 = 0$ 类光间隔，$\Delta s^2 < 0$ 类空间隔。
3. 间隔不变性：
$$\Delta s^2 = \Delta s'^2 \tag{3.2}$$
   这要求时空变换必须是线性变换。这个四维时空被称为闵氏时空。

3.1.2 Lorentz 变换

1. 四维时空点的位矢：
$$\boldsymbol{r} = (x_1, x_2, x_3, x_4) = (x, y, z, \mathrm{i}ct) \tag{3.3}$$
2. 四维时空点与原点之间的四维长度的平方：
$$\boldsymbol{r}^2 = x^2 + y^2 + z^2 + (\mathrm{i}ct)^2 = x^2 + y^2 + z^2 - c^2t^2 = -\Delta s^2 \tag{3.4}$$
3. $x - \mathrm{i}ct$ 平面内能保持四维长度不变的一般转动：
$$x' = x\cos\theta + \mathrm{i}ct\sin\theta, \quad y' = y, \quad z' = z, \quad \mathrm{i}ct' = -x\sin\theta + \mathrm{i}ct\cos\theta \tag{3.5}$$
4. 在 S 系中考察 S' 系的坐标原点的运动：
$$x' = vt\cos\theta + \mathrm{i}ct\sin\theta = 0, \quad \mathrm{i}ct' = -vt\sin\theta + \mathrm{i}ct\cos\theta \tag{3.6}$$
$$\tan\theta = -\frac{v}{\mathrm{i}c}, \quad \cos\theta = \frac{1}{\sqrt{1-\beta^2}}, \quad \sin\theta = -\frac{v}{\mathrm{i}c}\frac{1}{\sqrt{1-\beta^2}} \tag{3.7}$$
5. Lorentz 变换：
$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{3.8}$$
6. Lorentz 变换的矢量形式：
$$\boldsymbol{x} = \boldsymbol{x} + \frac{\gamma - 1}{\boldsymbol{\beta}^2}(\boldsymbol{\beta}\cdot\boldsymbol{x})\boldsymbol{\beta} - \gamma\boldsymbol{v}t, \quad t' = \gamma\left(t - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}\right) \tag{3.9}$$
7. 相对论速度变换：
$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}}, \quad u'_y = \frac{u_y}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)}, \quad u'_z = \frac{u_z}{\gamma\left(1 - \dfrac{u_x v}{c^2}\right)} \tag{3.10}$$
8. 相对论速度变换的矢量形式：
$$\boldsymbol{u}'_\parallel = \frac{\boldsymbol{u}_\parallel - \boldsymbol{v}}{1 - \boldsymbol{u}\cdot\boldsymbol{v}/c^2}, \quad \boldsymbol{u}'_\perp = \frac{\boldsymbol{u}_\perp}{\gamma\left(1 - \boldsymbol{v}\cdot\boldsymbol{u}/c^2\right)} \tag{3.11}$$

3.2 狭义相对论的四维形式

3.2.1 Lorentz 变换的四维形式

1. 四维时空坐标：

$$x_\mu = (x_1, x_2, x_3, x_4) = (\boldsymbol{x}, \mathrm{i}ct) \tag{3.12}$$

2. Lorentz 变换的矩阵形式：

$$\begin{pmatrix} x' \\ y' \\ z' \\ \mathrm{i}ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \mathrm{i}\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{i}\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ \mathrm{i}ct \end{pmatrix} \tag{3.13}$$

3. 简记为：

$$x'_\mu = L_{\mu\nu} x_\nu \tag{3.14}$$

4. 间隔不变性：

$$x'_\mu x'_\mu = L_{\mu\nu} x_\nu L_{\mu\lambda} x_\lambda = x_\nu x_\nu \tag{3.15}$$

5. Lorentz 变换是四维正交变换：

$$L_{\mu\nu} L_{\mu\lambda} = \delta_{\nu\lambda} \tag{3.16}$$

6. Lorentz 逆变换：

$$x_\mu = \delta_{\mu\nu} x_\nu = L_{\lambda\mu} L_{\lambda\nu} x_\nu = L_{\lambda\mu} x'_\lambda \tag{3.17}$$

3.2.2 相对论力学

1. 元间隔 $\mathrm{d}x_\mu \mathrm{d}x_\mu$：

$$\mathrm{d}x'_\mu \mathrm{d}x'_\mu = L_{\mu\nu} x_\nu L_{\mu\rho} \mathrm{d}x_\rho = \delta_{\nu\rho} \mathrm{d}x_\nu \mathrm{d}x_\rho = \mathrm{d}x_\nu \mathrm{d}x_\nu = -\mathrm{d}s^2 \tag{3.18}$$

2. 固有时 $\mathrm{d}\tau$：

$$\mathrm{d}s^2 = c^2 \mathrm{d}\tau^2 = c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \quad \Rightarrow \quad \mathrm{d}t = \gamma \mathrm{d}\tau \tag{3.19}$$

3. 四维速度：

$$u_\mu = \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} = (\gamma \boldsymbol{u}, \mathrm{i}c\gamma) \tag{3.20}$$

4. 四维动量：

$$p_\mu = m_0 u_\mu = \left(\gamma m_0 \boldsymbol{u}, \frac{\mathrm{i}}{c} \gamma m_0 c^2\right) = \left(\boldsymbol{p}, \frac{\mathrm{i}}{c} E\right) \tag{3.21}$$

5. 四维力：

$$K_\mu = \left(\gamma \boldsymbol{F}, \frac{\mathrm{i}}{c} \gamma \boldsymbol{F} \cdot \boldsymbol{u}\right) = \left(\boldsymbol{K}, \frac{\mathrm{i}}{c} \boldsymbol{K} \cdot \boldsymbol{u}\right) \tag{3.22}$$

6. 相对论动力学方程：

$$K_\mu = \frac{\mathrm{d}p_\mu}{\mathrm{d}\tau} \tag{3.23}$$

可以分为下面两个方程：

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t} (\gamma m_0 \boldsymbol{u}) = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}, \quad \boldsymbol{F} \cdot \boldsymbol{u} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\gamma m_0 c^2\right) = \frac{\mathrm{d}E}{\mathrm{d}t} \tag{3.24}$$

7. 相对论质量、动量、能量：

$$m = \gamma m_0, \quad \boldsymbol{p} = \gamma m_0 \boldsymbol{u}, \quad E = \gamma m_0 c^2 = T + m_0 c^2 \tag{3.25}$$

8. 相对论质量、动量、能量关系：

$$E^2 = p^2 c^2 + m_0^2 c^4 \tag{3.26}$$

3.3 电磁规律的相对论协变性

3.3.1 电动力学量的四维形式

1. 四维微商算符：

$$\partial_\mu = \frac{\partial}{\partial x_\mu} = \left(\nabla, \frac{1}{\mathrm{i}c}\frac{\partial}{\partial t}\right) \tag{3.27}$$

2. d'Alembert 算符：

$$\Box = \partial_\mu \partial_\mu = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{3.28}$$

3. 四维电流密度：

$$j_\mu = \rho_0 u_\mu = (\boldsymbol{j}, \mathrm{i}c\rho) \tag{3.29}$$

4. 四维势：

$$A_\mu = (\boldsymbol{A}, \frac{\mathrm{i}}{c}\varphi) \tag{3.30}$$

5. 电磁场张量：

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.31}$$

它是一个反对称张量，矩阵形式为

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -\frac{\mathrm{i}}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{\mathrm{i}}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{\mathrm{i}}{c}E_3 \\ \frac{\mathrm{i}}{c}E_1 & \frac{\mathrm{i}}{c}E_2 & \frac{\mathrm{i}}{c}E_3 & 0 \end{pmatrix} \tag{3.32}$$

6. 四维 Lorentz 力密度：

$$f_\mu = \rho_0 F_{\mu\nu} u_\nu = F_{\mu\nu} j_\nu = \left(\boldsymbol{f}, \frac{\mathrm{i}}{c}\boldsymbol{j}\cdot\boldsymbol{E}\right) \tag{3.33}$$

7. 四维电磁场能量动量张量：

$$T_{\mu\lambda} = \frac{1}{\mu_0}\left(F_{\mu\nu}F_{\nu\lambda} + \frac{1}{4}\delta_{\mu\lambda}F_{\nu\tau}F_{\nu\tau}\right) \tag{3.34}$$

它是一个无迹对称张量，即 $T_{\mu\lambda} = T_{\lambda\mu}$，$T_{\mu\mu} = 0$，矩阵形式为

$$T_{\mu\lambda} = \begin{pmatrix} -T_{11} & -T_{12} & -T_{13} & -\mathrm{i}cg_1 \\ -T_{21} & -T_{22} & -T_{23} & -\mathrm{i}cg_2 \\ -T_{31} & -t_{32} & -T_{33} & -\mathrm{i}cg_3 \\ -\frac{\mathrm{i}}{c}S_2 & -\frac{\mathrm{i}}{c}S_2 & \frac{\mathrm{i}}{c}S_3 & \omega \end{pmatrix} \tag{3.35}$$

3.3.2 相对论电动力学方程

1. 电荷守恒定律：

$$\partial_\mu j_\mu = 0 \tag{3.36}$$

2. Lorentz 规范：

$$\partial_\mu A_\mu = 0 \tag{3.37}$$

3. d'Alembert 方程：

$$\Box A_\mu = -\mu_0 j_\mu \tag{3.38}$$

4. Maxwell 方程：

$$\partial_\nu F_{\mu\nu} = \mu_0 j_\mu, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{3.39}$$

5. 能量动量守恒定律：

$$f_\mu = \partial_\lambda T_{\mu\lambda} \tag{3.40}$$

3.3.3　波矢的相对论变换

1. 四维波矢：
$$k_\mu = \left(\boldsymbol{k}, \frac{\mathrm{i}}{c}\omega\right) \tag{3.41}$$

2. 相位可以写成：
$$\varphi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t = k_\mu x_\mu \tag{3.42}$$

3. 相位不随参考系改变：$k'_\mu = L_{\mu\nu}k_\nu$
$$k'_x = \gamma\left(k_x - \frac{v}{c^2}\omega\right),\quad k'_y = k_y,\quad k'_z = k_z,\quad \omega' = \gamma(\omega - vk_x) \tag{3.43}$$

4. Doppler 效应与光行差公式：
$$\omega = \frac{\omega'}{\gamma(1-\beta\cos\theta)},\quad \tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta+\beta)} \tag{3.44}$$

3.3.4　电磁势和电磁场的相对论变换

1. 电磁势是四维协变矢量，按照 $A'_\mu = L_{\mu\nu}A_\nu$ 变换：
$$A'_x = \gamma\left(A_x - \frac{v}{c^2}\varphi\right),\quad A'_y = A_y,\quad A'_z = A_z,\quad \varphi' = \gamma(\varphi - vA_x) \tag{3.45}$$

2. 电磁场是四维协变张量，按照 $F'_{\mu\nu} = L_{\mu\lambda}L_{\nu\tau}F_{\lambda\tau}$ 变换：
$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - vB_z) \\ E'_z = \gamma(E_z + vB_y) \end{cases} \qquad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \frac{v}{c^2}E_z) \\ B'_z = \gamma(B_z - \frac{v}{c^2}E_y) \end{cases} \tag{3.46}$$

也可以写成与参考系相对运动方向垂直和平行的分量的形式：
$$\boldsymbol{E}'_\parallel = \boldsymbol{E}_\parallel,\quad \boldsymbol{E}'_\perp = \gamma(\boldsymbol{E}_\perp + \boldsymbol{v}\times\boldsymbol{B}_\perp),\quad \boldsymbol{B}'_\parallel = \boldsymbol{B}_\parallel,\quad \boldsymbol{B}'_\perp = \gamma(\boldsymbol{B}_\perp - \frac{\boldsymbol{v}}{c^2}\times\boldsymbol{E}_\perp) \tag{3.47}$$

3.3.5　包含电磁场的不变量

1. 两个 Lorentz 不变量：
$$B^2 - \frac{1}{c^2}E^2,\quad \boldsymbol{B}\cdot\boldsymbol{E} \tag{3.48}$$

2. 考察这两个标量的意义：对于真空中的电磁波有
$$|\boldsymbol{B}| = \frac{|\boldsymbol{E}|}{c},\quad \boldsymbol{B} = \frac{1}{c}\boldsymbol{e}_k\times\boldsymbol{E} \tag{3.49}$$

所以这两个不变量都为 0。说明在任何惯性系中，都有上面的关系成立。

Proof. 从电磁场张量出发可以构造出一个四维协变标量 $F_{\mu\nu}F_{\mu\nu}$

$$F'_{\mu\nu}F'_{\mu\nu} = L_{\mu\alpha}L_{\nu\beta}F_{\alpha\beta}L_{\mu\theta}L_{\nu\phi}F_{\theta\phi} = (L_{\mu\alpha}L_{\mu\theta})(L_{\nu\beta}L_{\nu\phi})F_{\alpha\beta}F_{\theta\phi} = F_{\alpha\beta}F_{\alpha\beta}$$

对于第一个不变量，可以这样构造
$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu} = B^2 - \frac{1}{c^2}E^2$$

对于第二个不变量，可以这样构造
$$\frac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\tau}F_{\mu\nu}F_{\lambda\tau} = \frac{1}{c}\boldsymbol{B}\cdot\boldsymbol{E}$$
□

3.4 电动力学的分析力学形式

3.4.1 电磁场中带电粒子的作用量

1. 自由带电粒子的作用量:

$$S_m = \int_1^2 \mathcal{L}_m \gamma \mathrm{d}\tau = \int_1^2 a \mathrm{d}\tau \tag{3.50}$$

2. 自由带电粒子的 Lagrange 量:

$$\mathcal{L}_m = \frac{a}{\gamma} = -\frac{m_0 c^2}{\gamma} \tag{3.51}$$

3. 带电粒子与电磁场相互作用的作用量:

$$S_{mf} = \int_1^2 \mathcal{L}_{mf} \gamma \mathrm{d}\tau = \int_1^2 b A_\mu \mathrm{d}x_\mu = \int_1^2 b A_\mu u_\mu \mathrm{d}\tau \tag{3.52}$$

4. 带电粒子与电磁场相互作用的 Lagrange 量:

$$\mathcal{L}_{mf} = \frac{b A_\mu u_\mu}{\gamma} = \frac{q A_\mu u_\mu}{\gamma} = -q\varphi + q\boldsymbol{v} \cdot \boldsymbol{A} \tag{3.53}$$

5. 电磁场中带电粒子的 Lagrange 量:

$$\mathcal{L}^{(m)} = \mathcal{L}_m + \mathcal{L}_{mf} = -\frac{m_0 c^2}{\gamma} - q(\varphi - \boldsymbol{v} \cdot \boldsymbol{A}) \tag{3.54}$$

6. 电磁场中带电粒子的正则动量:

$$\boldsymbol{P} = \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \boldsymbol{e}_i = \boldsymbol{p} + q\boldsymbol{A} \tag{3.55}$$

7. 电磁场中带电粒子的 Hamilton 量:

$$\mathcal{H} = \boldsymbol{P} \cdot \boldsymbol{v} - \mathcal{L} = \sqrt{(\boldsymbol{P} - q\boldsymbol{A}(\boldsymbol{r}, t))^2 c^2 + m_0^2 c^4} + q\varphi \tag{3.56}$$

3.4.2 电磁场的作用量

1. 自由电磁场的作用量:

$$S_f = \int_{\mathbf{R}^4} \widetilde{\mathcal{L}}_f \mathrm{d}\Omega = \int_{\mathbf{R}^4} c F_{\mu\nu} F_{\mu\nu} \mathrm{d}\Omega \tag{3.57}$$

2. 自由电磁场的 Lagrange 量密度:

$$\widetilde{\mathcal{L}}_f = c F_{\mu\nu} F_{\mu\nu} = \frac{\mathrm{i}}{4} \sqrt{\frac{\varepsilon_0}{\mu_0}} F_{\mu\nu} F_{\mu\nu} \tag{3.58}$$

3. 电磁场与带电粒子相互作用的作用量:

$$S_{mf} = \int_{\mathbf{R}^4} \widetilde{\mathcal{L}}_{mf} \mathrm{d}\Omega = \int_{\mathbf{R}^3} \rho \mathrm{d}V \int_1^2 A_\mu \mathrm{d}x_\mu = \int_{\mathbf{R}^4} A_\mu \rho \frac{\mathrm{d}x_\mu}{\mathrm{d}t} \mathrm{d}V \mathrm{d}t = \int_{\mathbf{R}^4} A_\mu j_\mu \frac{\mathrm{d}\Omega}{\mathrm{i}c} \tag{3.59}$$

4. 电磁场与带电粒子相互作用的 Lagrange 量密度:

$$\widetilde{\mathcal{L}}_{mf} = \frac{1}{\mathrm{i}c} A_\mu j_\mu \tag{3.60}$$

5. 电磁场的 Lagrange 量密度:

$$\widetilde{\mathcal{L}}^{(f)} = \widetilde{\mathcal{L}}_f + \widetilde{\mathcal{L}}_{mf} = \frac{\mathrm{i}}{4} \sqrt{\frac{\varepsilon_0}{\mu_0}} F_{\mu\nu} F_{\mu\nu} + \frac{1}{\mathrm{i}c} A_\mu j_\mu \tag{3.61}$$

3.4.3 带电粒子和电磁场的总作用量

$$S = S_m + S_{mf} + S_f = -m_0 c^2 \int_1^2 \mathrm{d}\tau + \frac{1}{\mathrm{i}c} \int_{\mathbf{R}^4} A_\mu j_\mu \mathrm{d}\Omega + \frac{\mathrm{i}}{4} \sqrt{\frac{\varepsilon_0}{\mu_0}} \int_{\mathbf{R}^4} F_{\mu\nu} F_{\mu\nu} \mathrm{d}\Omega \tag{3.62}$$

4 连续媒介质中的 Maxwell 方程组

4.1 介质中电磁场的性质

4.1.1 束缚电荷和分子电流

1. 分子的电偶极矩和电流磁矩:

$$\boldsymbol{p} = q\boldsymbol{l}, \quad \boldsymbol{m} = i\boldsymbol{a} \tag{4.1}$$

2. 介质的极化强度和磁化强度:

$$\boldsymbol{P}(\boldsymbol{r},t) = \lim_{\Delta V\to 0}\frac{\sum_i \boldsymbol{p}_i(t)}{\Delta V}, \quad \boldsymbol{M}(\boldsymbol{r},t) = \lim_{\Delta V\to 0}\frac{\sum_i \boldsymbol{m}_i(t)}{\Delta V} \tag{4.2}$$

3. 介质内的极化电荷密度和磁化电流密度:

$$\rho_p(\boldsymbol{r},t) = -\nabla\cdot\boldsymbol{P}(\boldsymbol{r},t), \quad \boldsymbol{j}_M(\boldsymbol{r},t) = \nabla\times\boldsymbol{M}(\boldsymbol{r},t) \tag{4.3}$$

4. 极化电流密度：电磁场变化引起极化电荷振动，形成极化电流

$$\boldsymbol{j}_p(\boldsymbol{r},t) = \frac{\partial \boldsymbol{P}(\boldsymbol{r},t)}{\partial t} \tag{4.4}$$

5. 介质内的电荷密度和电流密度:

$$\rho(\boldsymbol{r},t) = \rho_f(\boldsymbol{r},t) + \rho_p(\boldsymbol{r},t), \quad \boldsymbol{j}(\boldsymbol{r},t) = \boldsymbol{j}_f(\boldsymbol{r},t) + \boldsymbol{j}_M(\boldsymbol{r},t) + \boldsymbol{j}_p(\boldsymbol{r},t) \tag{4.5}$$

4.1.2 介质的电磁性质

1. 电位移矢量和磁场强度:

$$\boldsymbol{D}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}(\boldsymbol{r},t), \quad \boldsymbol{H}(\boldsymbol{r},t) = \frac{\boldsymbol{B}(\boldsymbol{r},t)}{\mu_0} - \boldsymbol{M}(\boldsymbol{r},t) \tag{4.6}$$

2. 在各向同性的线性介质中，有如下关系:

$$\boldsymbol{P}(\boldsymbol{r},t) = \chi_e\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t), \quad \boldsymbol{D}(\boldsymbol{r},t) = (1+\chi_e)\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) = \varepsilon_r\varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) = \varepsilon\boldsymbol{E}(\boldsymbol{r},t) \tag{4.7}$$

$$\boldsymbol{M}(\boldsymbol{r},t) = \chi_m\boldsymbol{H}(\boldsymbol{r},t), \quad \boldsymbol{B}(\boldsymbol{r},t) = (1+\chi_m)\mu_0\boldsymbol{H}(\boldsymbol{r},t) = \mu_r\mu_0\boldsymbol{H}(\boldsymbol{r},t) = \mu\boldsymbol{H}(\boldsymbol{r},t) \tag{4.8}$$

3. Ohm 定律：线性均匀介质导体的导电规律

$$\boldsymbol{j}_f(\boldsymbol{r},t) = \sigma\boldsymbol{E}(\boldsymbol{r},t) \tag{4.9}$$

4.1.3 介质中的 Maxwell 方程组

$$\begin{cases} \nabla\cdot\boldsymbol{D}(\boldsymbol{r},t) = \rho_f(\boldsymbol{r},t) \\ \nabla\times\boldsymbol{E}(\boldsymbol{r},t) = -\dfrac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \\ \nabla\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0 \\ \nabla\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{j}_f(\boldsymbol{r},t) + \dfrac{\partial\boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \end{cases} \tag{4.10}$$

$$\begin{cases} \displaystyle\oint_S \boldsymbol{D}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} = \int_V \rho_f(\boldsymbol{r},t)\mathrm{d}V \\ \displaystyle\oint_L \boldsymbol{E}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{B}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} \\ \displaystyle\oint_S \boldsymbol{B}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} = 0 \\ \displaystyle\oint_L \boldsymbol{H}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{l} = \int_S \boldsymbol{j}_f(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} + \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{D}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} \end{cases} \tag{4.11}$$

4.2 介质中电磁场的边界条件

1. 电磁场的边界条件:

$$\begin{aligned} \boldsymbol{e}_n \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \sigma_f, \quad \boldsymbol{e}_n \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0 \\ \boldsymbol{e}_n \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0, \quad \boldsymbol{e}_n \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{\alpha}_f \end{aligned} \tag{4.12}$$

2. 极化强度和磁化强度的边界条件:

$$\boldsymbol{e}_n \cdot (\boldsymbol{P}_2 - \boldsymbol{P}_1) = -\sigma_p, \quad \boldsymbol{e}_n \times (\boldsymbol{M}_2 - \boldsymbol{M}_1) = \boldsymbol{\alpha}_M \tag{4.13}$$

3. 电流密度的边界条件:

$$\boldsymbol{e}_n \cdot (\boldsymbol{j}_2 - \boldsymbol{j}_1) = -\frac{\partial \sigma}{\partial t} \tag{4.14}$$

4.3 介质中电磁场的能量守恒

1. 介质中电磁场能量守恒:

$$\frac{\partial w(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{S}(\boldsymbol{r},t) + \boldsymbol{j}_f(\boldsymbol{r},t) \cdot \boldsymbol{E}(\boldsymbol{r},t) = 0 \tag{4.15}$$

2. 能量密度:

$$w(\boldsymbol{r},t) = \frac{1}{2}\boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{D}(\boldsymbol{r},t) + \frac{1}{2}\boldsymbol{B}(\boldsymbol{r},t) \cdot \boldsymbol{H}(\boldsymbol{r},t) \tag{4.16}$$

3. 能流密度:

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t) \tag{4.17}$$

Proof. 区域 V 内的能量守恒方程应该写为

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V w(\boldsymbol{r},t)\mathrm{d}V = -\oint_S \boldsymbol{S}(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S} - \int_V \frac{\partial}{\partial t} w_{int}(\boldsymbol{r},t)\mathrm{d}V$$

其中，$w(\boldsymbol{r},t)$ 为能量密度，$\boldsymbol{S}(\boldsymbol{r},t)$ 为能流密度，最后一项 $w_{int}(\boldsymbol{r},t)$ 表示区域内场与粒子相互作用的能量交换，具体形式可以通过 Lorentz 力写出来

$$\frac{\partial}{\partial t} w_{int}(\boldsymbol{r},t) = \boldsymbol{f}(\boldsymbol{r},t) \cdot \boldsymbol{v} = \rho_f(\boldsymbol{r},t)\boldsymbol{v} \cdot \boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{j}_f(\boldsymbol{r},t) \cdot \boldsymbol{E}(\boldsymbol{r},t)$$

用 $\boldsymbol{E}(\boldsymbol{r},t)$ 点乘方程组(4.10)中的式四，可得

$$\boldsymbol{E}(\boldsymbol{r},t) \cdot (\nabla \times \boldsymbol{H}(\boldsymbol{r},t)) = \boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{j}_f(\boldsymbol{r},t) + \boldsymbol{E}(\boldsymbol{r},t) \cdot \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

再用 $\boldsymbol{H}(\boldsymbol{r},t)$ 点乘方程组(4.10)中的式二，可得

$$\boldsymbol{H}(\boldsymbol{r},t) \cdot (\nabla \times \boldsymbol{E}(\boldsymbol{r},t)) = -\boldsymbol{H}(\boldsymbol{r},t) \cdot \frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

利用式(1.20)给出

$$\begin{aligned} \nabla \cdot (\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)) &= \boldsymbol{H}(\boldsymbol{r},t) \cdot (\nabla \times \boldsymbol{E}(\boldsymbol{r},t)) - \boldsymbol{E}(\boldsymbol{r},t) \cdot (\nabla \times \boldsymbol{H}(\boldsymbol{r},t)) \\ &= -\boldsymbol{H}(\boldsymbol{r},t) \cdot \frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} - \boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{j}_f(\boldsymbol{r},t) - \boldsymbol{E}(\boldsymbol{r},t) \cdot \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \\ &= -\boldsymbol{j}_f(\boldsymbol{r},t) \cdot \boldsymbol{E}(\boldsymbol{r},t) - \frac{1}{2}\left(\varepsilon \frac{\partial E^2(\boldsymbol{r},t)}{\partial t} + \frac{1}{\mu}\frac{\partial B^2(\boldsymbol{r},t)}{\partial t}\right) \end{aligned}$$

若令

$$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t)$$

$$w(\boldsymbol{r},t) = \frac{1}{2}\left(\varepsilon E^2(\boldsymbol{r},t) + \frac{1}{\mu}B^2(\boldsymbol{r},t)\right) = \frac{1}{2}\boldsymbol{E}(\boldsymbol{r},t) \cdot \boldsymbol{D}(\boldsymbol{r},t) + \frac{1}{2}\boldsymbol{B}(\boldsymbol{r},t) \cdot \boldsymbol{H}(\boldsymbol{r},t)$$

此式又可以写为

$$\frac{\partial w(\boldsymbol{r},t)}{\partial t} + \nabla \cdot \boldsymbol{S}(\boldsymbol{r},t) + \boldsymbol{j}_f(\boldsymbol{r},t) \cdot \boldsymbol{E}(\boldsymbol{r},t) = 0$$

□

5 静电场

5.1 静电场的边值问题

5.1.1 静电场的基本方程

1. 静电场中 Maxwell 方程退化为：

$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho_f(\boldsymbol{r}), \quad \nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \tag{5.1}$$

2. 在各向同性的线性介质中：

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon \boldsymbol{E}(\boldsymbol{r}) \tag{5.2}$$

3. 静电学问题简化为在一定的边界条件下求解以下方程：

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_f(\boldsymbol{r})}{\varepsilon}, \quad \nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \tag{5.3}$$

4. 连续媒介质存在时的 Possion 方程：

$$\nabla^2 \varphi(\boldsymbol{r}) = -\frac{\rho_f(\boldsymbol{r})}{\varepsilon} \tag{5.4}$$

5. Laplace 方程：

$$\nabla^2 \varphi(\boldsymbol{r}) = 0 \tag{5.5}$$

6. 无界空间中静电势的解：

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' \tag{5.6}$$

5.1.2 静电场中导体的边界条件

1. 第一类边界条件：

$$\varphi_1 = \varphi_2 \tag{5.7}$$

2. 第二类边界条件：

$$\varepsilon_2 \frac{\partial \varphi_2}{\partial n} - \varepsilon_1 \frac{\partial \varphi_1}{\partial n} = -\sigma_f \tag{5.8}$$

5.2 唯一性定理

设区域 V 内介质部分的体电荷密度 $\rho_f(\boldsymbol{r})$ 和面电荷密度 $\sigma_f(\boldsymbol{r})$ 给定，导体部分的总电荷量 Q_f 给定，在边界面上 $\varphi(\boldsymbol{r})|_{\partial V}$ 或 $\left.\frac{\partial \varphi(\boldsymbol{r})}{\partial n}\right|_{\partial V}$ 给定，那么静电学问题的解是唯一的。

Proof. 先考虑区域 V 内的媒介质为同质的情况。假设有两组解 $\varphi(\boldsymbol{r}), \psi(\boldsymbol{r})$ 均是满足边界条件的解，令

$$\varphi(\boldsymbol{r}) = \varphi'(\boldsymbol{r}) - \varphi''(\boldsymbol{r})$$

其满足 Laplace 方程

$$\nabla^2 \varphi(\boldsymbol{r}) = \nabla^2 \varphi'(\boldsymbol{r}) - \nabla^2 \varphi''(\boldsymbol{r}) = 0$$

并且在边界上，下面两个式子总有至少一个成立

$$\varphi(\boldsymbol{r})|_S = \varphi'(\boldsymbol{r})|_S - \varphi''(\boldsymbol{r})|_S = 0, \quad \left.\frac{\partial \varphi(\boldsymbol{r})}{\partial n}\right|_S = \left.\frac{\partial \varphi'(\boldsymbol{r})}{\partial n}\right|_S - \left.\frac{\partial \varphi''(\boldsymbol{r})}{\partial n}\right|_S = 0$$

考虑以下积分

$$0 = \int_V \varepsilon \varphi(\boldsymbol{r}) \nabla^2 \varphi(\boldsymbol{r}) \mathrm{d}V = \oint_S \varepsilon \varphi(\boldsymbol{r}) \left(\nabla \varphi(\boldsymbol{r})\right) \cdot \mathrm{d}\boldsymbol{S} - \int_V \varepsilon \left(\nabla \varphi(\boldsymbol{r})\right)^2 \mathrm{d}V = -\int_V \varepsilon \left(\nabla \varphi(\boldsymbol{r})\right)^2 \mathrm{d}V$$

也就是说

$$\nabla \varphi(\boldsymbol{r}) = 0 \;\Rightarrow\; \varphi(\boldsymbol{r}) = \varphi'(\boldsymbol{r}) - \varphi''(\boldsymbol{r}) = \mathrm{Const}$$

这说明在 V 内，$\varphi(\boldsymbol{r})$ 是一个常数，而静电势函数附加上一个常数对电场强度分布并无影响。

再考虑区域 V 是由不同质的媒介质和导体组成的情况。对每一导体部分我们有

$$\nabla^2\varphi_i(\boldsymbol{r}) = \nabla^2\varphi_i'(\boldsymbol{r}) - \nabla^2\varphi_i''(\boldsymbol{r}) = 0$$

考虑以下积分

$$0 = \sum_{i=1}^{n}{}' \int_{V_i} \varepsilon_i\varphi_i(\boldsymbol{r})\nabla^2\varphi_i(\boldsymbol{r})\mathrm{d}V_i = \sum_{i=1}^{n}{}' \int_{S_i} \varepsilon_i\varphi_i(\boldsymbol{r})\left(\nabla\varphi_i(\boldsymbol{r})\right)\cdot\mathrm{d}\boldsymbol{S}_i - \sum_{i=1}^{n}{}' \int_{V_i} \varepsilon_i\left(\nabla\varphi_i(\boldsymbol{r})\right)^2\mathrm{d}V_i$$

其中，$\sum'$ 表示求和时去除导体占据的区域。下面分两种情况论证上式右侧第一项为 0：

1. 当 S_i 的一部分为 ∂V 的一部分时，边界条件给出 $\varphi(\boldsymbol{r})|_{\partial V} = 0$ 或 $\left.\dfrac{\partial\varphi(\boldsymbol{r})}{\partial n}\right|_{\partial V} = 0$，于是显然为 0。
2. 当 S_i 的一部分被 V_i 和 V_j 共享时，电势的连续性给出 $\varphi_i(\boldsymbol{r}) = \varphi_j(\boldsymbol{r})$，再分两种情况：
   (1) 当分界面两边皆为介质时，考虑下面两个积分的贡献之和

$$\begin{aligned}
\Delta_{ij} &= \int_{S_i'} \varepsilon_i\varphi_i(\boldsymbol{r})\left(\nabla\varphi_i(\boldsymbol{r})\right)\cdot\mathrm{d}\boldsymbol{S}_i + \int_{S_j'} \varepsilon_j\varphi_j(\boldsymbol{r})\left(\nabla\varphi_j(\boldsymbol{r})\right)\cdot\mathrm{d}\boldsymbol{S}_j \\
&= \int_{S_i'} \varepsilon_i\varphi_i(\boldsymbol{r})\left(\nabla\varphi_i(\boldsymbol{r})\right)\cdot\mathrm{d}\boldsymbol{S}_i + \int_{S_j'} \varepsilon_j\varphi_i(\boldsymbol{r})\left(\nabla\varphi_j(\boldsymbol{r})\right)\cdot(-\mathrm{d}\boldsymbol{S}_i) \\
&= \int_{S_i'} \varphi_i(\boldsymbol{r})\left(\varepsilon_i\frac{\partial\varphi_i(\boldsymbol{r})}{\partial n} - \varepsilon_j\frac{\partial\varphi_j(\boldsymbol{r})}{\partial n}\right)\mathrm{d}S_i \\
&= \int_{S_i'} \varphi_i(\boldsymbol{r})\left[\left(\varepsilon_i\frac{\partial\varphi_i'(\boldsymbol{r})}{\partial n} - \varepsilon_j\frac{\partial\varphi_j'(\boldsymbol{r})}{\partial n}\right) - \left(\varepsilon_i\frac{\partial\varphi_i''(\boldsymbol{r})}{\partial n} - \varepsilon_j\frac{\partial\varphi_j''(\boldsymbol{r})}{\partial n}\right)\right]\mathrm{d}S_i \\
&= \int_{S_i'} \varphi_i(\boldsymbol{r})\left[-\sigma_{fi}^{\varphi'}(\boldsymbol{r}) - \left(-\sigma_{fi}^{\varphi''}(\boldsymbol{r})\right)\right]\mathrm{d}S_i = 0
\end{aligned}$$

   (2) 当分界面两边分别为介质和导体时，导体为等势体将给出

$$\begin{aligned}
&\sum_{i=1}^{n}{}' \int_{S_i} \varepsilon_i\varphi_i(\boldsymbol{r})\left(\nabla\varphi_i(\boldsymbol{r})\right)\cdot\mathrm{d}\boldsymbol{S}_i = \sum_{i=1}^{n}{}'\varphi \int_{S_i'} \varepsilon_i\frac{\partial\varphi_i(\boldsymbol{r})}{\partial n}\mathrm{d}S = \varphi\sum_{i=1}^{n}{}' \int_{S_i'} \varepsilon_i\left(\frac{\partial\varphi_i'(\boldsymbol{r})}{\partial n} - \frac{\partial\varphi_i''(\boldsymbol{r})}{\partial n}\right)\mathrm{d}S \\
&= \varphi\sum_{i=1}^{n}{}' \int_{S_i'} \left(\sigma_{fi}^{\varphi''} - \sigma_{fi}^{\varphi'}\right)\mathrm{d}S = \varphi\left(\sum_{i=1}^{n}{}' \int_{S_i'} \sigma_{fi}^{\varphi''}\mathrm{d}S - \sum_{i=1}^{n}{}' \int_{S_i'} \sigma_{fi}^{\varphi'}\mathrm{d}S\right) = \varphi(Q_f - Q_f) = 0
\end{aligned}$$

因此，我们得到

$$\sum_{i=1}^{n}{}' \int_{V_i} \varepsilon_i\left(\nabla\varphi_i(\boldsymbol{r})\right)^2\mathrm{d}V_i = 0$$

如前所述，它隐含着 Possion 方程的两个解之间最多差一个常数。故静电学的唯一性定理成立。 □

5.3 分离变量法

5.3.1 球坐标系下的分离变量法

1. 球坐标系下的 Laplace 方程：

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\varphi^2} = 0 \tag{5.9}$$

2. 分离变量：

$$\varphi(r,\theta,\varphi) = R(r)Y(\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi) \tag{5.10}$$

3. 当轴对称时：与 φ 无关，$m = 0$

$$\varphi(r,\theta) = \sum_{l=0}^{+\infty}\left(C_l r^l + D_l r^{-(l+1)}\right)P_l(\cos\theta) \tag{5.11}$$

4. 当非轴对称时：

$$\varphi(r,\theta,\varphi) = \sum_{l=0}^{+\infty}\sum_{m=-l}^{l}(C_{lm}r^l + D_{lm}r^{-(l+1)})Y_{lm}(\theta,\varphi) \tag{5.12}$$

5.3.2 柱坐标系下的分离变量法

1. 柱坐标系下的 Laplace 方程：

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\varphi^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \tag{5.13}$$

2. 分离变量：

$$\varphi(\rho,\varphi,z)=R(\rho)\varPhi(\varphi)Z(z) \tag{5.14}$$

3. 当 $\lambda=0$ 时：

$$\varphi(\rho,\varphi,z)=\sum_{m=-\infty}^{+\infty}C_m\rho^m\mathrm{e}^{\mathrm{i}m\varphi}(C+Dz) \tag{5.15}$$

4. 当 $\lambda>0$ 时：

   (1) 当轴对称时：与 φ 无关，$m=0$

$$\varphi(\rho,z)=\sum_{n=1}^{+\infty}J_0\left(\frac{x_n^{(0)}}{b}\rho\right)\left[A_n\mathrm{e}^{\frac{x_n^{(0)}}{b}z}+B_n\mathrm{e}^{-\frac{x_n^{(0)}}{b}z}\right] \tag{5.16}$$

   (2) 当非轴对称时：

$$\varphi(\rho,\varphi,z)=\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{+\infty}J_m\left(\frac{x_n^{(m)}}{b}\rho\right)\left[A_n^{(m)}\mathrm{e}^{\frac{x_n^{(m)}}{b}z}+B_n^{(m)}\mathrm{e}^{-\frac{x_n^{(m)}}{b}z}\right]\mathrm{e}^{\mathrm{i}m\varphi} \tag{5.17}$$

5.4 镜像法

1. 无限大接地导体平面：

$$q'=-q,\quad x=a \tag{5.18}$$

2. 半径为 r 的接地球壳：

$$q'=-\frac{r}{a}q,\quad x=\frac{r^2}{a} \tag{5.19}$$

5.5 Green 函数法

$$\nabla_r^2G(\boldsymbol{r},\boldsymbol{r}')=-\frac{1}{\varepsilon_0}\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{5.20}$$

5.5.1 三种区域的 Green 函数

1. 无界空间中的 Green 函数：

$$G_0(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi\varepsilon_0}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}=\frac{1}{4\pi\varepsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{5.21}$$

2. 上半平面的 Green 函数：

$$G_1(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right) \tag{5.22}$$

3. 球外空间的 Green 函数：

$$G_2(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\alpha}}-\frac{1}{\sqrt{\left(\frac{rr'}{R_0}\right)^2+R_0^2-2rr'\cos\alpha}}\right) \tag{5.23}$$

5.5.2 电势边值问题的解

1. 区域 V 内 Possion 方程的解：

$$\varphi(\boldsymbol{r})=\frac{\varepsilon_0}{\varepsilon}\int_V\rho_f(\boldsymbol{r}')G(\boldsymbol{r}',\boldsymbol{r})\mathrm{d}V'+\varepsilon_0\oint_S\left(G(\boldsymbol{r},\boldsymbol{r}')\frac{\partial\varphi(\boldsymbol{r}')}{\partial n'}-\varphi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r}',\boldsymbol{r})}{\partial n'}\right)\mathrm{d}S' \tag{5.24}$$

2. 对于第一类边值问题，$G(\boldsymbol{r}',\boldsymbol{r})|_S=0$，电势简化为：

$$\varphi(\boldsymbol{r})=\frac{\varepsilon_0}{\varepsilon}\int_V\rho_f(\boldsymbol{r}')G(\boldsymbol{r}',\boldsymbol{r})\mathrm{d}V'-\varepsilon_0\oint_S\varphi(\boldsymbol{r}')\frac{\partial G(\boldsymbol{r}',\boldsymbol{r})}{\partial n'}\mathrm{d}S' \tag{5.25}$$

3. 对于第二类边值问题，$\left.\dfrac{\partial G(\boldsymbol{r}',\boldsymbol{r})}{\partial n'}\right|_{\boldsymbol{r}'\in S}=-\dfrac{1}{\varepsilon_0 S}$，电势简化为：

$$\varphi(\boldsymbol{r})=\frac{\varepsilon_0}{\varepsilon}\int_V\rho_f(\boldsymbol{r}')G(\boldsymbol{r}',\boldsymbol{r})\mathrm{d}V'+\varepsilon_0\oint_S G(\boldsymbol{r}',\boldsymbol{r})\frac{\partial\varphi(\boldsymbol{r}')}{\partial n'}\mathrm{d}S'+\frac{1}{S}\oint_S\varphi(\boldsymbol{r}')\mathrm{d}S' \tag{5.26}$$

5.6 电势多级展开

5.6.1 直角坐标系中电势多级展开

1. 考虑远场电势，作如下级数展开：

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}-\boldsymbol{r}'\cdot\nabla\frac{1}{r}+\frac{1}{2!}x_i'x_j'\partial_i\partial_j\frac{1}{r}+\cdots,\quad r\gg r' \tag{5.27}$$

2. 电势展开为：

$$\begin{aligned}\varphi(\boldsymbol{r})&=\frac{1}{4\pi\varepsilon}\int_{V'}\rho_f(\boldsymbol{r}')\left(\frac{1}{r}-\boldsymbol{r}'\cdot\nabla\frac{1}{r}+\frac{1}{2}x_i'x_j'\partial_i\partial_j\frac{1}{r}+\cdots\right)\mathrm{d}V'\\&=\frac{1}{4\pi\varepsilon}\left(\frac{Q}{r}+\frac{\boldsymbol{P}\cdot\boldsymbol{r}}{r^3}+\frac{1}{6}\mathcal{D}_{ij}\partial_i\partial_j\frac{1}{r}+\cdots\right)\end{aligned} \tag{5.28}$$

3. 电荷系统的总电荷：

$$Q=\int_V\rho_f(\boldsymbol{r}')\mathrm{d}V' \tag{5.29}$$

4. 电荷系统的电偶极矩：

$$\boldsymbol{P}=\int_V\rho(\boldsymbol{r}')\boldsymbol{r}'\mathrm{d}V' \tag{5.30}$$

5. 电荷系统的电四极矩分量：

$$\mathcal{D}_{ij}=\int_V 3x_i'x_j'\rho_f(\boldsymbol{r}')\mathrm{d}V' \tag{5.31}$$

6. 电四极矩的另一种定义：

$$\mathcal{D}_{ij}=\int_V(3x_i'x_j'-r'^2\delta_{ij})\rho_f(\boldsymbol{r}')\mathrm{d}V' \tag{5.32}$$

是一个对称无迹 (traceless) 矩阵，满足 $\mathrm{tr}\left(\overleftrightarrow{\mathcal{D}}\right)=0$。因此，独立分量个数有 $9-3-1=5$ 个。

7. 电偶极矩产生的电势：

$$\varphi^{(1)}(\boldsymbol{r})=\frac{1}{4\pi\varepsilon}\frac{\boldsymbol{P}\cdot\boldsymbol{r}}{r^3} \tag{5.33}$$

8. 电偶极矩产生的电场：

$$\boldsymbol{E}(\boldsymbol{r})=-\nabla\varphi^{(1)}=\frac{1}{4\pi\varepsilon}\left(\frac{3(\boldsymbol{P}\cdot\boldsymbol{r})}{r^5}-\frac{\boldsymbol{P}}{r^3}\right) \tag{5.34}$$

5.6.2 球坐标系中电势多级展开

1. 将远场的电势展开为球谐函数：

$$\varphi(\boldsymbol{r})=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'=\frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi),\quad r\gg r' \tag{5.35}$$

2. 同时利用 Legendre 函数的完备性，考虑球谐函数的加法原理，作出以下展开：

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(\cos\gamma)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r'^l}{r^{l+1}}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi) \tag{5.36}$$

3. 比较以上两式得到多级矩：

$$q_{lm}=\int_{V'}Y^*_{lm}(\theta',\varphi')r'^l\rho(\boldsymbol{r}')\mathrm{d}V' \tag{5.37}$$

4. 将积分在直接坐标中写出：

$$q_{00}=\frac{1}{\sqrt{4\pi}}\int_{V'}\rho(\boldsymbol{r}')\mathrm{d}V'=\frac{1}{\sqrt{4\pi}}Q \tag{5.38}$$

$$q_{10}=\sqrt{\frac{3}{4\pi}}P_z \tag{5.39}$$

$$q_{11}=-\sqrt{\frac{3}{8\pi}}(P_x-\mathrm{i}P_y) \tag{5.40}$$

$$q_{20}=\frac{1}{2}\sqrt{\frac{5}{4\pi}}\mathcal{D}_{33} \tag{5.41}$$

$$q_{21}=-\frac{1}{3}\sqrt{\frac{15}{8\pi}}(\mathcal{D}_{13}-\mathrm{i}\mathcal{D}_{23}) \tag{5.42}$$

$$q_{22}=\frac{1}{12}\sqrt{\frac{15}{2\pi}}(\mathcal{D}_{11}-2\mathrm{i}\mathcal{D}_{12}-\mathcal{D}_{22}) \tag{5.43}$$

5.7 静电场的能量

1. 电荷体系的总静电能：

$$W=\frac{1}{2}\int_{\mathbf{R}^3}\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{D}(\boldsymbol{r})\mathrm{d}V=\frac{1}{2}\int_V\rho_f(\boldsymbol{r})\varphi(\boldsymbol{r})\mathrm{d}V \tag{5.44}$$

2. 外电场对带电体的静电能：

$$W_i=\int_V\rho(\boldsymbol{r})\varphi_e(\boldsymbol{r})\mathrm{d}V \tag{5.45}$$

3. 当电荷分布于小区域时，外电场对带电体的静电能在原点的级数展开式为：

$$W_i=Q\varphi_e(0)+\boldsymbol{P}\cdot\nabla\varphi_e(0)+\frac{1}{6}\mathcal{D}_{ij}\partial_i\partial_j\varphi_e(0) \tag{5.46}$$

(1) 第二项解释为电偶极矩在外电场中的静电能，又可以写为：

$$W_i^{(1)}=\boldsymbol{P}\cdot\nabla\varphi_e(0)=-\boldsymbol{P}\cdot\boldsymbol{E}_e(0) \tag{5.47}$$

(2) 第三项解释为电四极矩在外电场中的静电能，又可以写为：

$$W_i^{(2)}=\frac{1}{6}\mathcal{D}_{ij}\partial_i\partial_j\varphi_e(0)=-\frac{1}{6}\mathcal{D}_{ij}\partial_iE_j^{(e)}(0) \tag{5.48}$$

4. 电偶极矩感受到的力：均匀外场

$$\boldsymbol{F}=-\nabla W^{(1)}=\nabla(\boldsymbol{P}\cdot\boldsymbol{E}_e(0))=\boldsymbol{P}\cdot\nabla\boldsymbol{E}_e(0) \tag{5.49}$$

5. 电偶极矩感受到的力矩：均匀外场

$$\boldsymbol{M}_e=-\frac{\partial W}{\partial\theta}\boldsymbol{e}_{\delta\theta}=-PE_e(0)\sin\theta\boldsymbol{e}_{\delta\theta}=\boldsymbol{P}\times\boldsymbol{E}_e(0) \tag{5.50}$$

6 静磁场

6.1 静磁场的边值问题

6.1.1 静磁场的基本方程

1. 静磁场中 Maxwell 方程退化为:

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0, \quad \nabla \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{j}_f(\boldsymbol{r}) \tag{6.1}$$

2. 在各向同性的线性介质中:

$$\boldsymbol{B}(\boldsymbol{r}) = \mu \boldsymbol{H}(\boldsymbol{r}) \tag{6.2}$$

3. 静磁学问题简化为在一定的边界条件下求解以下方程:

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0, \quad \nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu \boldsymbol{j}_f(\boldsymbol{r}) \tag{6.3}$$

4. 引入磁矢势 $\boldsymbol{A}(\boldsymbol{r})$，第一个方程自动满足:

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) \tag{6.4}$$

5. 选取 Coulomb 规范 $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$，第二个方程变为磁矢势的 Possion 方程:

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}) = \nabla^2 \boldsymbol{A}(\boldsymbol{r}) - \nabla(\nabla \cdot \boldsymbol{A}(\boldsymbol{r})) = -\nabla \times (\nabla \times \boldsymbol{A}(\boldsymbol{r})) = -\mu \boldsymbol{j}_f(\boldsymbol{r}) \tag{6.5}$$

6. 无界空间中的磁矢势的解:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \int_V \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' \tag{6.6}$$

7. 计算对应的磁感应强度可得 Bio-Savart 定律:

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) = \nabla \times \frac{\mu}{4\pi} \int_V \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}V' \tag{6.7}$$

$$= \frac{\mu}{4\pi} \int_V \left(\nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) \times \boldsymbol{j}(\boldsymbol{r}') \mathrm{d}V' = \frac{\mu}{4\pi} \int_V \boldsymbol{j}(\boldsymbol{r}') \times \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \mathrm{d}V' \tag{6.8}$$

6.1.2 静磁场的边界条件

1. 第一类边界条件:

$$\boldsymbol{A}_1 = \boldsymbol{A}_2 \tag{6.9}$$

2. 第二类边界条件:

$$\boldsymbol{e}_n \times \left(\frac{1}{\mu_2} \nabla \times \boldsymbol{A}_2 - \frac{1}{\mu_1} \nabla \times \boldsymbol{A}_1 \right) = \boldsymbol{\alpha}_f \tag{6.10}$$

6.2 磁标势法

1. 处理空间无自由电流分布的问题:

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = 0 \tag{6.11}$$

2. 仿照静电学引入磁标势 $\varphi_m(\boldsymbol{r})$:

$$\boldsymbol{H}(\boldsymbol{r}) = -\nabla \varphi_m(\boldsymbol{r}) \tag{6.12}$$

3. 在永久铁磁物质中有如下关系:

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0, \quad \boldsymbol{B}(\boldsymbol{r}) = \mu_0 (\boldsymbol{H}(\boldsymbol{r}) + \boldsymbol{M}(\boldsymbol{r})) \tag{6.13}$$

4. 变形可得:

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{M}(\boldsymbol{r}) \tag{6.14}$$

5. 束缚磁荷密度:

$$\rho_m(\boldsymbol{r}) = -\mu_0 \nabla \cdot \boldsymbol{M}(\boldsymbol{r}) \tag{6.15}$$

6. (6.14)可以写为:

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = \frac{\rho_m(\boldsymbol{r})}{\mu_0} \tag{6.16}$$

7. 磁标势的 Possion 方程:

$$\nabla^2 \varphi_m(\boldsymbol{r}) = -\frac{\rho_m(\boldsymbol{r})}{\mu_0} \tag{6.17}$$

8. 边界条件:

$$\varphi_{m1} = \varphi_{m2} \tag{6.18}$$

$$\mu_2 \frac{\partial \varphi_{m2}}{\partial n} = \mu_1 \frac{\partial \varphi_{m1}}{\partial n} \tag{6.19}$$

表 1: 静电势与磁标势公式对比

静电场	静磁场（无自由电流）
$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0$	$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = 0$
$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = (\rho_f(\boldsymbol{r}) + \rho_p(\boldsymbol{r}))/\varepsilon_0$	$\nabla \cdot \boldsymbol{H}(\boldsymbol{r}) = \rho_m(\boldsymbol{r})/\mu_0$
$\rho_p(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{P}(\boldsymbol{r})$	$\rho_m(\boldsymbol{r}) = -\mu_0 \nabla \cdot \boldsymbol{M}(\boldsymbol{r})$
$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r})$	$\boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{H}(\boldsymbol{r}) + \mu_0 \boldsymbol{M}(\boldsymbol{r})$
$\boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r})$	$\boldsymbol{H}(\boldsymbol{r}) = -\nabla \varphi_m(\boldsymbol{r})$
$\nabla^2 \varphi(\boldsymbol{r}) = -(\rho_f(\boldsymbol{r}) + \rho_p(\boldsymbol{r}))/\varepsilon_0$	$\nabla^2 \varphi_m(\boldsymbol{r}) = -\rho_m(\boldsymbol{r})/\mu_0$

6.3 磁矢势多级展开

6.3.1 直角坐标系中磁矢势多级展开

1. 考虑远场磁矢势，只作出前两项展开:

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \frac{1}{r} - \boldsymbol{r}' \cdot \nabla \frac{1}{r} + \cdots = \frac{1}{r} + \frac{\boldsymbol{r}' \cdot \boldsymbol{r}}{r^3} + \cdots, \quad r \gg r' \tag{6.20}$$

2. 磁矢势展开为:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \frac{1}{r} \int_V \boldsymbol{j}(\boldsymbol{r}') \mathrm{d}V' + \frac{\mu}{4\pi} \frac{\boldsymbol{r}}{r^3} \cdot \int_V \boldsymbol{r}' \boldsymbol{j}(\boldsymbol{r}') \mathrm{d}V' = 0 + \frac{\mu}{4\pi} \frac{\boldsymbol{r}}{r^3} \cdot \int_V \boldsymbol{r}' \boldsymbol{j}(\boldsymbol{r}') \mathrm{d}V' = \frac{\mu}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \tag{6.21}$$

3. 电流系统的总磁偶极矩:

$$\boldsymbol{m} = \frac{1}{2} \int_V \boldsymbol{r}' \times \boldsymbol{j}(\boldsymbol{r}') \mathrm{d}V' \tag{6.22}$$

4. 磁偶极矩产生的磁感应强度:

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{4\pi} \nabla \times \left(\frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3} \right) = \frac{\mu}{4\pi} \left(\frac{3(\boldsymbol{m} \cdot \boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{m}}{r^3} \right) \tag{6.23}$$

5. 磁偶极矩产生的磁标势:

$$\varphi_m^{(1)}(\boldsymbol{r}) = \frac{\boldsymbol{m} \cdot \boldsymbol{r}}{4\pi r^3} \tag{6.24}$$

6.3.2 球坐标系中磁矢势多级展开

1. 球坐标下多级展开到前两项：

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}=\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^{l}P_l(\cos\gamma)=\frac{1}{r}\left(1+\frac{r'}{r}\cos\gamma\right)+\cdots \tag{6.25}$$

2. 代入磁矢势表达式：

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu}{4\pi r}\int_{V'}\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V'+\frac{\mu}{4\pi r}\int_{V'}\frac{r'}{r}\cos\gamma\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V' \tag{6.26}$$

3. 注意到第一项即 $\boldsymbol{A}^{(0)}(\boldsymbol{r})=0$，故：

$$\boldsymbol{A}(\boldsymbol{r})=\frac{\mu}{4\pi r}\int_{V'}\frac{r'}{r}\cos\gamma\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V'=\frac{\mu}{4\pi}\frac{\boldsymbol{r}}{r^3}\cdot\int_{V'}\boldsymbol{r}'\boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V'=\frac{\mu}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3} \tag{6.27}$$

6.4 静磁场的能量

1. 电流体系的总静磁能量：

$$W=\frac{1}{2}\int_{\mathbf{R}^3}\boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{H}(\boldsymbol{r})\mathrm{d}V=\frac{1}{2}\int_{V}\boldsymbol{j}_f(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r})\mathrm{d}V \tag{6.28}$$

2. 外磁场对电流的静磁能：

$$W_i=\int_V\boldsymbol{j}_f(\boldsymbol{r})\cdot\boldsymbol{A}_e(\boldsymbol{r})\mathrm{d}V \tag{6.29}$$

3. 将上式应用到一个载有电流 I 的线圈上，可得

$$W_i=I\oint_L\boldsymbol{A}_e(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l}=\int_S(\nabla\times\boldsymbol{A}_e(\boldsymbol{r}))\cdot\mathrm{d}\boldsymbol{S}=I\int_S\boldsymbol{B}_e(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S} \tag{6.30}$$

当线圈的线度远小于外电流源 $\boldsymbol{j}_e(\boldsymbol{r})$ 到线圈的距离时，将 $\boldsymbol{B}_e(\boldsymbol{r})$ 在 0 点附近展开为

$$\boldsymbol{B}_e(\boldsymbol{r})=\boldsymbol{B}_e(0)+\boldsymbol{r}\cdot\nabla\boldsymbol{B}_e(0)+\cdots \tag{6.31}$$

于是线圈在外场中的静磁能的第一项就写为

$$W_i^{(0)}=I\int_S\boldsymbol{B}_e(0)\cdot\mathrm{d}\boldsymbol{S}=\boldsymbol{m}\cdot\boldsymbol{B}_e(0) \tag{6.32}$$

在这样的假设下，电流管实际上已经成为了磁偶极子模型。

4. 与电偶极子不同，考虑电磁感应过程后，外磁场中磁偶极子的势能应写为：

$$U(\boldsymbol{r})=-\boldsymbol{m}\cdot\boldsymbol{B}_e(0) \tag{6.33}$$

5. 磁偶极矩感受到的力：均匀外场

$$\boldsymbol{F}=-\nabla U(\boldsymbol{r})=\nabla(\boldsymbol{m}\cdot\boldsymbol{B}_e(0))=\boldsymbol{m}\cdot\nabla\boldsymbol{B}_e(0) \tag{6.34}$$

6. 磁偶极矩感受到的力矩：均匀外场

$$\boldsymbol{M}=-\frac{\partial U(\boldsymbol{r})}{\partial\theta}\boldsymbol{e}_{\delta\theta}=-mB_e(0)\sin\theta\boldsymbol{e}_{\delta\theta}=\boldsymbol{m}\times\boldsymbol{B}_e(0) \tag{6.35}$$

7 电磁波的传播

7.1 平面电磁波

7.1.1 无色散介质中的单色平面波

1. 考虑无源的 Maxwell 方程组:

$$\nabla(\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t))-\nabla^2\boldsymbol{E}(\boldsymbol{r},t)=-\nabla^2\boldsymbol{E}(\boldsymbol{r},t)=-\nabla\times\frac{\partial\boldsymbol{B}}{\partial t}=-\mu\varepsilon\frac{\partial^2\boldsymbol{E}(\boldsymbol{r},t)}{\partial t^2}\tag{7.1}$$

2. 电磁场波动方程:

$$\frac{\partial^2\boldsymbol{E}(\boldsymbol{r},t)}{\partial t^2}-\frac{1}{\mu\varepsilon}\nabla^2\boldsymbol{E}(\boldsymbol{r},t)=0\tag{7.2}$$

$$\frac{\partial^2\boldsymbol{H}(\boldsymbol{r},t)}{\partial t^2}-\frac{1}{\mu\varepsilon}\nabla^2\boldsymbol{H}(\boldsymbol{r},t)=0\tag{7.3}$$

3. 波速:

$$u=\frac{1}{\sqrt{\mu\varepsilon}}<\frac{1}{\sqrt{\mu_0\varepsilon_0}}=c\tag{7.4}$$

4. 电磁场的单色平面波解:

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)},\quad \boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{H}_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\tag{7.5}$$

5. 横波条件:

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t)=0\Rightarrow\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{r},t)=0\tag{7.6}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},t)=0\Rightarrow\boldsymbol{k}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0\tag{7.7}$$

6. 磁感应强度:

$$\boldsymbol{B}(\boldsymbol{r},t)=-\frac{\mathrm{i}}{\omega}\nabla\times\boldsymbol{E}(\boldsymbol{r},t)=\frac{1}{\omega}\boldsymbol{k}\times\boldsymbol{E}(\boldsymbol{r},t)=\sqrt{\mu\varepsilon}\boldsymbol{e}_k\times\boldsymbol{E}(\boldsymbol{r},t)\tag{7.8}$$

$$\left|\frac{\boldsymbol{E}(\boldsymbol{r},t)}{\boldsymbol{B}(\boldsymbol{r},t)}\right|=\frac{1}{\sqrt{\mu\varepsilon}}=u\tag{7.9}$$

7.1.2 无色散介质中单色平面波的能量

1. 能量密度:

$$w(\boldsymbol{r},t)=\frac{1}{2}\left(\varepsilon\mathrm{Re}E^2(\boldsymbol{r},t)+\frac{1}{\mu}\mathrm{Re}B^2(\boldsymbol{r},t)\right)=\varepsilon\mathrm{Re}E^2(\boldsymbol{r},t)=\frac{1}{\mu}\mathrm{Re}B^2(\boldsymbol{r},t)\tag{7.10}$$

2. 能流密度:

$$\boldsymbol{S}(\boldsymbol{r},t)=\mathrm{Re}\boldsymbol{E}(\boldsymbol{r},t)\times\mathrm{Re}\boldsymbol{H}(\boldsymbol{r},t)=\sqrt{\frac{\varepsilon}{\mu}}\mathrm{Re}E^2(\boldsymbol{r},t)\boldsymbol{e}_k=uw(\boldsymbol{r},t)\boldsymbol{e}_k\tag{7.11}$$

3. 平均能量密度:

$$\langle w(\boldsymbol{r})\rangle=\frac{1}{T}\int_0^T w(\boldsymbol{r},t)\mathrm{d}t=\frac{1}{T}\int_0^T\varepsilon E_0^2\cos^2(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\mathrm{d}t=\frac{1}{2}\varepsilon E_0^2\tag{7.12}$$

4. 平均能流密度:

$$\langle\boldsymbol{S}(\boldsymbol{r})\rangle=\frac{1}{T}\int_0^T\boldsymbol{S}(\boldsymbol{r},t)\mathrm{d}t=\frac{1}{T}\int_0^T\sqrt{\frac{\varepsilon}{\mu}}E_0^2\cos^{(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\boldsymbol{e}_k\mathrm{d}t=\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}E_0^2\boldsymbol{e}_k\tag{7.13}$$

7.1.3 有色散介质中的单色平面波

1. 有色散介质中有关系:

$$\boldsymbol{D}_\omega(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}_\omega(\boldsymbol{r}, t) + \varepsilon_0 \int_0^{+\infty} f(\tau) \boldsymbol{E}_\omega(\boldsymbol{r}, t-\tau) \mathrm{d}\tau \tag{7.14}$$

$$\boldsymbol{B}_\omega(\boldsymbol{r}, t) = \mu_0 \boldsymbol{H}_\omega(\boldsymbol{r}, t) + \mu_0 \int_0^{+\infty} g(\tau) \boldsymbol{H}(\boldsymbol{r}, t-\tau) \mathrm{d}\tau \tag{7.15}$$

2. 与无色散介质中关系有相同形式:

$$\boldsymbol{D}_\omega(\boldsymbol{r}, t) = \varepsilon_0 (1 + \widetilde{f}(\omega)) \boldsymbol{E}_\omega(\boldsymbol{r}, t) = \varepsilon(\omega) \boldsymbol{E}_\omega(\boldsymbol{r}, t) \tag{7.16}$$

$$\boldsymbol{B}_\omega(\boldsymbol{r}, t) = \mu_0 (1 + \widetilde{g}(\omega)) \boldsymbol{H}_\omega(\boldsymbol{r}, t) = \mu(\omega) \boldsymbol{H}_\omega(\boldsymbol{r}, t) \tag{7.17}$$

3. Helmholtz 方程:

$$\nabla^2 \boldsymbol{E}_\omega(\boldsymbol{r}) + k^2 \boldsymbol{E}_\omega(\boldsymbol{r}) = 0, \quad k^2(\omega) = \mu(\omega)\varepsilon(\omega)\omega^2 \tag{7.18}$$

4. 横场条件:

$$\nabla \cdot \boldsymbol{E}_\omega(\boldsymbol{r}) = 0 \tag{7.19}$$

5. 相速度:

$$u(\omega) = \frac{\omega}{k} = \frac{1}{\sqrt{\mu(\omega)\varepsilon(\omega)}} \tag{7.20}$$

6. 电场的一个解:

$$\boldsymbol{E}_\omega(\boldsymbol{r}) = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \tag{7.21}$$

7. 磁感应强度:

$$\boldsymbol{B}_\omega(\boldsymbol{r}) = -\frac{\mathrm{i}}{\omega} \nabla \times \boldsymbol{E}_\omega(\boldsymbol{r}) \tag{7.22}$$

7.2 电磁波在介质面上的折射和反射

1. 两个独立的边界条件:

$$\boldsymbol{e}_n \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0, \quad \boldsymbol{e}_n \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{\alpha}_f \tag{7.23}$$

2. 入射波、反射波和折射波:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}, \quad \boldsymbol{E}'(\boldsymbol{r}, t) = \boldsymbol{E}'_0 \mathrm{e}^{(\boldsymbol{k}'\cdot\boldsymbol{r} - \omega \mathrm{t})}, \quad \boldsymbol{E}''(\boldsymbol{r}, t) = \boldsymbol{E}''_0 \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \tag{7.24}$$

7.2.1 反射和折射定律

1. 边界条件:

$$\boldsymbol{e}_n \times \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(k_x x + k_y y)} + \boldsymbol{e}_n \times \boldsymbol{E}'_0 \mathrm{e}^{\mathrm{i}(k'_x x + k'_y y)} + \boldsymbol{e}_n \times \boldsymbol{E}''_0 \mathrm{e}^{\mathrm{i}(k''_x x + k''_y y)} \tag{7.25}$$

2. x 和 y 坐标的任意性要求:

$$k_x = k'_x = k''_x, \quad k_y = k'_y = k''_y = 0 \tag{7.26}$$

3. 界面上波矢的 x 分量相等:

$$k \sin\theta = k' \sin\theta' = k'' \sin\theta'' \tag{7.27}$$

4. 反射定律:

$$\theta' = \theta \tag{7.28}$$

5. 折射定律:

$$n_1 \sin\theta = n_2 \sin\theta'' \tag{7.29}$$

7.2.2 Fresnel 公式

1. 边界条件:

$$\boldsymbol{e}_n \times \boldsymbol{E}_0 + \boldsymbol{e}_n \times \boldsymbol{E}_0' = \boldsymbol{e}_n \times \boldsymbol{E}_0'' \tag{7.30}$$

$$\boldsymbol{e}_n \times \left(\frac{1}{\mu_1(\omega)}\boldsymbol{k} \times \boldsymbol{E}_0\right) + \boldsymbol{e}_n \times \left(\frac{1}{\mu_1(\omega)}\boldsymbol{k}' \times \boldsymbol{E}_0'\right) = \boldsymbol{e}_n \times \left(\frac{1}{\mu_2(\omega)}\boldsymbol{k}'' \times \boldsymbol{E}_0''\right) \tag{7.31}$$

2. $\boldsymbol{E}_0 \perp$ 入射面:
   (1) 边界条件:

$$E_0 + E_0' = E_0'' \tag{7.32}$$

$$\sqrt{\varepsilon_1(\omega)}(E_0 - E_0')\cos\theta = \sqrt{\varepsilon_2(\omega)}E_0''\cos\theta'' \tag{7.33}$$

   (2) 振幅关系:

$$\frac{E_\perp'}{E_\perp} = \frac{\sin(\theta-\theta'')}{\sin(\theta+\theta'')}, \quad \frac{E_\perp''}{E_\perp} = \frac{2\cos\theta\sin\theta''}{\sin(\theta+\theta'')} \tag{7.34}$$

3. $\boldsymbol{E}_0 \parallel$ 入射面:
   (1) 边界条件:

$$E_0\cos\theta - E_0'\cos\theta = E_0''\cos\theta'' \tag{7.35}$$

$$\sqrt{\varepsilon_1(\omega)}(E_0 + E_0') = \sqrt{\varepsilon_2(\omega)}E_0'' \tag{7.36}$$

   (2) 振幅关系:

$$\frac{E_\parallel'}{E_\parallel} = \frac{\tan(\theta-\theta'')}{\tan(\theta+\theta'')}, \quad \frac{E_\parallel''}{E_\parallel} = \frac{2\cos\theta\sin\theta''}{\sin(\theta+\theta'')\cos(\theta-\theta'')} \tag{7.37}$$

7.2.3 全反射

1. 界面上波矢的 x 分量相等:

$$k_x'' = k_x = k\sin\theta \tag{7.38}$$

2. 折射波波矢的 z 分量:

$$k_z'' = \sqrt{k''^2 - k_x''^2} = k\sqrt{n_{21}^2 - \sin^2\theta} = \mathrm{i}k\sqrt{\sin^2\theta - n_{21}^2} = \mathrm{i}\kappa \tag{7.39}$$

3. 穿透深度:

$$\kappa^{-1} = \frac{1}{k\sqrt{\sin^2\theta - n_{21}^2}} = \frac{\lambda_0}{2\pi n_1\sqrt{\sin^2\theta - n_{21}^2}} \tag{7.40}$$

4. 折射波的电磁场:

$$\boldsymbol{E}''(\boldsymbol{r},t) = E_0''\mathrm{e}^{-\kappa z}\mathrm{e}^{\mathrm{i}(k_x''x-\omega t)}\boldsymbol{e}_y \tag{7.41}$$

$$\boldsymbol{H}''(\boldsymbol{r},t) = \frac{1}{\omega\mu_2(\omega)}\left(k_x''E_0''\mathrm{e}^{-\kappa z}\mathrm{e}^{\mathrm{i}(k_x''x-\omega t)}\boldsymbol{e}_z - \mathrm{i}\kappa E_0''\mathrm{e}^{-\kappa z}\mathrm{e}^{\mathrm{i}(k_x''x-\omega t)}\boldsymbol{e}_x\right) \tag{7.42}$$

5. 折射波的平均能流密度:

$$\langle \boldsymbol{S}''(\boldsymbol{r})\rangle = \frac{1}{T}\int_0^T \mathrm{Re}\boldsymbol{E}''(\boldsymbol{r},t) \times \mathrm{Re}\boldsymbol{H}''(\boldsymbol{r},t)\mathrm{d}t = \frac{1}{2}\sqrt{\frac{\varepsilon_2(\omega)}{\mu_2(\omega)}}E_0''^2\mathrm{e}^{-2\kappa z}\frac{\sin\theta}{n_{21}}\boldsymbol{e}_x \tag{7.43}$$

6. 反射波的振幅和相位:

$$E' = \frac{\cos\theta - \mathrm{i}\sqrt{\sin^2\theta - n_{21}^2}}{\cos\theta + \mathrm{i}\sqrt{\sin^2\theta - n_{21}^2}}E_0 = \mathrm{e}^{-2\mathrm{i}\phi}E_0 \tag{7.44}$$

$$\tan\phi = \frac{\sqrt{\sin^2\theta - n_{21}^2}}{\cos\theta} \tag{7.45}$$

7.3 导体内电磁波的传播

7.3.1 导体内的自由电荷分布

1. 导体内的电荷密度方程：

$$\frac{\partial \rho_f(\boldsymbol{r},t)}{\partial t} = -\nabla \cdot \boldsymbol{j}(r,t) = -\sigma \nabla \cdot \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\sigma}{\varepsilon}\rho_f(\boldsymbol{r},t) \tag{7.46}$$

2. 导体内的电荷密度：

$$\rho_f(\boldsymbol{r},t) = \rho_f(\boldsymbol{r},0)\mathrm{e}^{-\frac{\sigma}{\varepsilon}t} \tag{7.47}$$

3. 良导体条件：导体内部自由电荷为 0

$$\nu = \frac{1}{T} \ll \frac{1}{\tau} \Rightarrow \frac{\sigma}{\omega\varepsilon(\omega)} \gg 1 \tag{7.48}$$

7.3.2 导体内的单色平面波

1. Maxwell 方程组式四引入传导电流项：

$$\nabla \times \boldsymbol{H}_\omega(\boldsymbol{r}) = \sigma \boldsymbol{E}_\omega(\boldsymbol{r}) - \mathrm{i}\omega\varepsilon(\omega)\boldsymbol{E}_\omega(\boldsymbol{r}) = -\mathrm{i}\omega\varepsilon'(\omega)\boldsymbol{E}_\omega(\boldsymbol{r}) \tag{7.49}$$

2. 复电容率：

$$\varepsilon'(\omega) = \varepsilon(\omega) + \mathrm{i}\frac{\sigma}{\varepsilon} \tag{7.50}$$

3. 导体内的 Helmholtz 方程：

$$\nabla^2 \boldsymbol{E}_\omega(\boldsymbol{r}) + k^2 \boldsymbol{E}_\omega(\boldsymbol{r}) = 0, \quad k^2 = \mu(\omega)\varepsilon'(\omega)\omega^2 \tag{7.51}$$

4. 复波矢：

$$\boldsymbol{k} = \boldsymbol{\beta} + \mathrm{i}\boldsymbol{\alpha} \tag{7.52}$$

$$\boldsymbol{\beta}^2 - \boldsymbol{\alpha}^2 = \omega^2\mu(\omega)\varepsilon(\omega), \quad \boldsymbol{\beta}\cdot\boldsymbol{\alpha} = \frac{1}{2}\omega\mu(\omega)\sigma \tag{7.53}$$

5. 导体内单色平面波的电磁场：

$$\boldsymbol{E}_\omega(\boldsymbol{r},t) = \boldsymbol{E}_0\mathrm{e}^{-\boldsymbol{\alpha}\cdot\boldsymbol{r}}\mathrm{e}^{\mathrm{i}(\boldsymbol{\beta}\cdot\boldsymbol{r}-\omega t)} \tag{7.54}$$

$$\boldsymbol{B}_\omega(\boldsymbol{r},t) = \frac{1}{\omega}(\boldsymbol{\beta} + \mathrm{i}\boldsymbol{\alpha}) \times \boldsymbol{E}_\omega(\boldsymbol{r},t) \tag{7.55}$$

7.3.3 趋肤效应

1. 正入射折射波的复波矢：良导体条件

$$\beta = \omega\sqrt{\frac{\mu(\omega)\varepsilon(\omega)}{2}}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2(\omega)}}+1\right)^{\frac{1}{2}} \approx \sqrt{\frac{\omega\mu(\omega)\sigma}{2}} \tag{7.56}$$

$$\alpha = \omega\sqrt{\frac{\mu(\omega)\varepsilon(\omega)}{2}}\left(\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2(\omega)}}-1\right)^{\frac{1}{2}} \approx \sqrt{\frac{\omega\mu(\omega)\sigma}{2}} \tag{7.57}$$

2. 正入射折射波的电场：

$$\boldsymbol{E}''(\boldsymbol{r},t) = \boldsymbol{E}_0''\boldsymbol{e}^{-\alpha z}\boldsymbol{e}^{\mathrm{i}(\beta z-\omega t)} \tag{7.58}$$

3. 穿透深度：

$$z_c = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu(\omega)\sigma}} \tag{7.59}$$

4. 正入射折射波的磁场：

$$\boldsymbol{H}''(\boldsymbol{r},t)=\frac{1}{\omega\mu(\omega)}(\beta+\mathrm{i}\alpha)\boldsymbol{e}_z\times\boldsymbol{E}''(\boldsymbol{r},t)=\sqrt{\frac{\sigma}{\omega\mu(\omega)}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\boldsymbol{e}_z\times\boldsymbol{E}''(\boldsymbol{r},t) \tag{7.60}$$

5. 导体内能量主要是磁场能：

$$\frac{W_B(\boldsymbol{r},t)}{W_E(\boldsymbol{r},t)}=\frac{\mu(\omega)}{\varepsilon(\omega)}\left|\frac{\boldsymbol{H}_0''}{\boldsymbol{E}_0''}\right|^2=\frac{\mu(\omega)}{\varepsilon(\omega)}\frac{\sigma}{\omega\mu(\omega)}=\frac{\sigma}{\omega\varepsilon(\omega)}\gg 1 \tag{7.61}$$

6. 边界条件：

$$E_0+E_0'=E_0'',\quad k(E_0-E_0')=k''E_0'' \tag{7.62}$$

7. 正入射反射波的电场振幅：

$$E_0'=\frac{k-k''}{k+k''}E_0=-\frac{1+\mathrm{i}-\sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}}{1+\mathrm{i}+\sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}}E_0 \tag{7.63}$$

8. 正入射的反射系数：

$$\mathcal{R}=\left|\frac{E_0'}{E_0}\right|^2=\frac{\left(1-\sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}\right)^2+1}{\left(1+\sqrt{\dfrac{2\omega\varepsilon_0}{\sigma}}\right)^2+1}\approx 1-2\sqrt{\frac{2\omega\varepsilon_0}{\sigma}} \tag{7.64}$$

7.4 谐振腔

1. 谐振腔中的 Helmholtz 方程：

$$\nabla^2u(x,y,z)+k^2u(x,y,z)=0 \tag{7.65}$$

2. 分离变量：

$$u(x,y,z)=X(x)Y(y)Z(z) \tag{7.66}$$

$$X''+k_x^2X=0,\quad Y''+k_y^2Y=0,\quad Z''+k_z^2Z=0 \tag{7.67}$$

$$k_x^2+k_y^2+k_z^2=k^2=\omega^2\mu(\omega)\varepsilon(\omega) \tag{7.68}$$

3. 边界条件：

$$E_y=E_z=0,\quad \frac{\partial E_x}{\partial x}=0\quad (x=0,\ l_1) \tag{7.69}$$

$$E_x=E_z=0,\quad \frac{\partial E_y}{\partial y}=0\quad (y=0,\ l_2) \tag{7.70}$$

$$E_x=E_y=0,\quad \frac{\partial E_z}{\partial z}=0\quad (z=0,\ l_3) \tag{7.71}$$

4. 谐振腔内电场的驻波解：

$$E_x(x,y,z,t)=A_1\cos k_xx\sin k_yy\sin k_zz\mathrm{e}^{-\mathrm{i}\omega t} \tag{7.72}$$

$$E_y(x,y,z,t)=A_2\sin k_xx\cos k_yy\sin k_zz\mathrm{e}^{-\mathrm{i}\omega t} \tag{7.73}$$

$$E_z(x,y,z,t)=A_3\sin k_xx\sin k_yy\cos k_zz\mathrm{e}^{-\mathrm{i}\omega t} \tag{7.74}$$

$$k_x=\frac{n_1\pi}{l_1},\quad k_y=\frac{n_2\pi}{l_2},\quad k_z=\frac{n_3\pi}{l_3},\quad n_1,n_2,n_3=0,1,2,\ldots \tag{7.75}$$

5. 横波条件：

$$\nabla\cdot\boldsymbol{E}=A_1k_x+A_2k_y+A_3k_z=0 \tag{7.76}$$

6. 谐振腔的本征频率：

$$\omega_{n_1n_2n_3}=\frac{\pi}{\sqrt{\mu(\omega)\varepsilon(\omega)}}\sqrt{\left(\frac{n_1}{l_1}\right)^2+\left(\frac{n_2}{l_2}\right)^2+\left(\frac{n_3}{l_3}\right)^2}\geqslant\omega_{110} \tag{7.77}$$

7.5 波导

1. 波导中的 Helmholtz 方程:

$$\nabla^2 u(x,y)+(k^2-k_z)^2 u(x,y)=0 \tag{7.78}$$

2. 分离变量:

$$u(x,y)=X(x)Y(y) \tag{7.79}$$

$$X''+k_xX^2=0,\quad Y''+k_yY^2=0 \tag{7.80}$$

$$k_x^2+k_y^2+k_z^2=k^2=\omega^2\mu(\omega)\varepsilon(\omega) \tag{7.81}$$

3. 边界条件:

$$E_y=E_z=0,\quad \frac{\partial E_x}{\partial x}=0\quad (x=0,\ a) \tag{7.82}$$

$$E_x=E_z=0,\quad \frac{\partial E_y}{\partial y}=0\quad (y=0,\ b) \tag{7.83}$$

4. 波导内电场的驻波解:

$$E_x(x,y,z,t)=A_1\cos k_xx\sin k_yy\mathrm{e}^{\mathrm{i}(k_zz-\omega t)} \tag{7.84}$$

$$E_y(x,y,z,t)=A_2\sin k_xx\cos k_yy\mathrm{e}^{\mathrm{i}(k_zz-\omega t)} \tag{7.85}$$

$$E_z(x,y,z,t)=A_3\sin k_xx\sin k_yy\mathrm{e}^{\mathrm{i}(k_zz-\omega t)} \tag{7.86}$$

$$k_x=\frac{m\pi}{a},\quad k_y=\frac{n\pi}{b},\quad m,n=0,1,2,\dots \tag{7.87}$$

5. 横波条件:

$$\nabla\cdot\boldsymbol{E}=A_1k_x+A_2k_y-\mathrm{i}A_3k_z=0 \tag{7.88}$$

6. 波导的本征频率:

$$\omega_{mn}=\frac{1}{\sqrt{\mu(\omega)\varepsilon(\omega)}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2+k_z^2}\geqslant\frac{\pi}{\sqrt{\mu(\omega)\varepsilon(\omega)}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}\geqslant\omega_{10} \tag{7.89}$$

7. TE_{10} 波的电磁场分量:

$$E_x(x,y,z)=0,\quad E_y(x,y,z)=\frac{\mathrm{i}\omega\mu(\omega)a}{\pi}H_0\sin\frac{\pi x}{a}\mathrm{e}^{\mathrm{i}k_zz},\quad E_z(x,y,z)=0 \tag{7.90}$$

$$H_x(x,y,z)=-\frac{\mathrm{i}k_za}{\pi}H_0\sin\frac{\pi x}{a},\quad H_y(x,y,z)=0,\quad H_z(x,y,z)=H_0\cos\frac{\pi x}{a}\mathrm{e}^{\mathrm{i}k_zz} \tag{7.91}$$

8. 面电流密度:

$$\boldsymbol{\alpha}_{fa}(x,y,z)=\boldsymbol{e}_y\times\boldsymbol{H}(x,y,z)=\frac{\mathrm{i}k_za}{\pi}H_0\sin\frac{\pi x}{a}\boldsymbol{e}_z-H_0\cos\frac{\pi x}{a}\mathrm{e}^{\mathrm{i}k_zz}\boldsymbol{e}_x \tag{7.92}$$

$$\boldsymbol{\alpha}_{fb}(x,y,z)=\boldsymbol{e}_x\times\boldsymbol{H}(x,y,z)=-H_0\cos\frac{\pi x}{a}\mathrm{e}^{\mathrm{i}k_zz}\boldsymbol{e}_y \tag{7.93}$$

7.6 等离子体

7.6.1 等离子体的 Coulomb 屏蔽

1. 热平衡下点电荷电势的 Possion 方程:

$$\nabla^2\varphi(\boldsymbol{r})=\frac{1}{\varepsilon_0}(-Zen_i(\boldsymbol{r})+en_e(\boldsymbol{r})-q\delta(\boldsymbol{r})) \tag{7.94}$$

2. 热平衡下电子分布服从 Boltzmann 分布:

$$n_e(\boldsymbol{r})=n_{e0}\exp\left(\frac{e\varphi(\boldsymbol{r})}{k_BT}\right)=n_{e0}\left(1+\frac{e\varphi(\boldsymbol{r})}{k_BT}\right) \tag{7.95}$$

3. 整体电中性:

$$Zen_i - en_{e0} = 0 \tag{7.96}$$

4. Possion 方程变为:

$$\left(\nabla^2 - \frac{1}{\lambda^2}\right)\varphi(\boldsymbol{r}) = -\frac{q}{\varepsilon_0}\delta(\boldsymbol{r}), \quad \lambda^2 = \frac{k_B T\varepsilon_0}{n_{e0}e^2} \tag{7.97}$$

5. 屏蔽 Coulomb 势:

$$\varphi(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0 r}\boldsymbol{e}^{-\frac{r}{\lambda}} \tag{7.98}$$

7.6.2 等离子体振荡

1. 电子的连续性方程:

$$\frac{\partial n(\boldsymbol{r},t)}{\partial t} + \nabla\cdot(n(\boldsymbol{r},t)\boldsymbol{v}(\boldsymbol{r},t)) = 0 \tag{7.99}$$

2. 电子动力学方程:

$$m\left(\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + (\nabla\cdot\boldsymbol{v}(\boldsymbol{r},t))\boldsymbol{v}(\boldsymbol{r},t)\right) = -e\boldsymbol{E}(\boldsymbol{r},t) \tag{7.100}$$

3. 内电场:

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{\varepsilon_0}\delta\rho(\boldsymbol{r},t) = -\frac{e}{\varepsilon_0}(n(\boldsymbol{r},t)-n_0) \tag{7.101}$$

4. 微扰下给出: $\delta\rho = -e(n(\boldsymbol{r},t)-n_0) = 0$, $\boldsymbol{v}(\boldsymbol{r},t) = 0$

$$\frac{\partial^2\delta\rho(\boldsymbol{r},t)}{\partial t^2} + \frac{n_0e^2}{m\varepsilon_0}\delta\rho(\boldsymbol{r},t) = 0 \tag{7.102}$$

$$\delta\rho(\boldsymbol{r},t) = \delta\rho(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega_p t}, \quad \omega_p = \sqrt{\frac{n_0e^2}{m\varepsilon_0}} \tag{7.103}$$

7.6.3 等离子体内电磁波的传播

1. 外电场的影响是引起电子运动的附加速度:

$$m\frac{\partial \boldsymbol{v}_e(\boldsymbol{r},t)}{\partial t} = -e\boldsymbol{E}_e(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t} \tag{7.104}$$

$$\boldsymbol{v}_e(\boldsymbol{r}) = -\frac{\mathrm{i}e}{m\omega}\boldsymbol{E}_e(\boldsymbol{r}) \tag{7.105}$$

2. 外电场引起的电流密度:

$$\boldsymbol{j}_e(\boldsymbol{r}) = -n_0e\boldsymbol{v}_e(\boldsymbol{r}) = \sigma(\omega)\boldsymbol{E}_e(\boldsymbol{r}), \quad \sigma(\omega) = \frac{\mathrm{i}n_0e^2}{m\omega} \tag{7.106}$$

3. 等离子体的复电容率:

$$\varepsilon'(\omega) = \varepsilon(\omega) + \mathrm{i}\frac{\sigma(\omega)}{\omega} \approx \varepsilon_0 - \frac{n_0e^2}{m\omega} \tag{7.107}$$

4. 等离子体内电磁波的波数:

$$k = \omega\sqrt{\mu(\omega)\varepsilon'(\omega)} \approx \omega\sqrt{\mu_0\varepsilon_0\left(1-\frac{n_0e^2}{m\omega^2\varepsilon_0}\right)} \tag{7.108}$$

5. 等离子体的折射率:

$$n = \sqrt{1-\frac{n_0e^2}{m\omega^2\varepsilon_0}} = \sqrt{1-\frac{\omega_p^2}{\omega^2}} \tag{7.109}$$

8 电磁波的辐射

8.1 d'Alembert 方程和推迟势

1. 电磁势：

$$\boldsymbol{B}(\boldsymbol{r},t)=\nabla\times\boldsymbol{A}(\boldsymbol{r},t),\quad \boldsymbol{E}(\boldsymbol{r},t)=-\nabla\varphi(\boldsymbol{r},t)-\frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}\tag{8.1}$$

2. Lorentz 规范：

$$\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t)+\frac{1}{c^2}\frac{\partial\varphi(\boldsymbol{r},t)}{\partial t}=0\tag{8.2}$$

3. d'Alembert 方程：

$$\nabla^2\boldsymbol{A}(\boldsymbol{r},t)-\frac{1}{c^2}\frac{\partial^2\boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2}=-\mu_0\boldsymbol{j}_f(\boldsymbol{r},t)\tag{8.3}$$

$$\nabla^2\varphi(\boldsymbol{r},t)-\frac{1}{c^2}\frac{\partial^2\varphi(\boldsymbol{r},t)}{\partial t^2}=-\frac{1}{\varepsilon_0}\rho_f(\boldsymbol{r},t)\tag{8.4}$$

4. 推迟势：

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\int_V\frac{\boldsymbol{j}_f\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\tag{8.5}$$

$$\varphi(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho_f\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\tag{8.6}$$

Proof. 采用 Landau《场论》8.1 节的方法。假设一个无穷小电荷位于原点 $\rho_f(\boldsymbol{r},t)=q(t)\delta(\boldsymbol{r})$。在 $\boldsymbol{r}\neq 0$ 处，考虑到空间旋转不变性，电标势与 θ,φ 无关，d'Alembert 方程为

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial t}\right)-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=0$$

作变量代换

$$\varphi(r,t)=\frac{u(r,t)}{r}\Rightarrow\frac{\partial^2 u}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0.$$

行波法给出上式的通解为

$$u(r,t)=f\left(t-\frac{r}{c}\right)+g\left(t+\frac{r}{c}\right)$$

第一项代表发散波，第二项代表会聚波。在电磁辐射问题中，我们只关心发散波，因此电标势可写为

$$\varphi(r,t)=\frac{f\left(t-\frac{r}{c}\right)}{r}$$

当 $r\to 0$ 时，$\varphi(r,t)$ 的解将趋向于静电势的解，忽略时变项可得

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)\approx-\frac{1}{\varepsilon_0}q(t)\delta(\boldsymbol{r})$$

显然这是一个 Possion 方程，考虑到推迟性的解为

$$\varphi(r,t)\approx\frac{1}{4\pi\varepsilon_0}\frac{q(t')}{r}=\frac{1}{4\pi\varepsilon_0}\frac{q\left(t-\frac{r}{c}\right)}{r}$$

所以

$$f(t-\frac{r}{c})=\frac{1}{4\pi\varepsilon_0}q\left(t-\frac{r}{c}\right)$$

所以位于 $\boldsymbol{r}'$ 处电荷引发的空间各点的电标势为

$$\varphi(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}'|}Q\left(\boldsymbol{r}',t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right)$$

对于磁矢势可以类似处理。

下面证明上面的两个推迟势解满足 Lorentz 规范。

$$
\begin{aligned}
\nabla\cdot\boldsymbol{A}(\boldsymbol{r},t) &= \frac{\mu_0}{4\pi}\int_V \nabla_r\cdot\frac{\boldsymbol{j}(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \left(\boldsymbol{j}(\boldsymbol{r}',t')\cdot\nabla_r\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}+\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\partial}{\partial t'}\boldsymbol{j}(\boldsymbol{r}',t')\nabla_r t'\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \left(-\boldsymbol{j}(\boldsymbol{r}',t')\cdot\nabla_{r'}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\partial}{\partial t'}\boldsymbol{j}(\boldsymbol{r}',t')\nabla_{r'} t'\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \left(-\nabla_{r'}\cdot\frac{\boldsymbol{j}(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|}+\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla_{r'}\boldsymbol{j}(\boldsymbol{r}',t')|_{t'\text{不变}}\right)\mathrm{d}V' \\
&= \frac{\mu_0}{4\pi}\int_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla_{r'}\cdot\boldsymbol{j}(\boldsymbol{r}',t')|_{t'\text{不变}}\mathrm{d}V'
\end{aligned}
$$

$$
\frac{\partial\varphi(\boldsymbol{r},t)}{\partial t} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{\partial\rho(\boldsymbol{r}',t')}{\partial t'}\mathrm{d}V'
$$

于是

$$
\nabla_r\cdot\boldsymbol{A}(\boldsymbol{r},t)+\frac{1}{c^2}\frac{\partial\varphi(\boldsymbol{r},t)}{\partial t} = \frac{\mu_0}{4\pi}\int_V \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\left(\nabla_{r'}\cdot\boldsymbol{j}(\boldsymbol{r}',t')|_{t'\text{不变}}+\frac{\partial\rho(\boldsymbol{r}',t')}{\partial t'}\right)\mathrm{d}V' = 0 \qquad \square
$$

8.2 天线辐射

8.2.1 天线产生的电磁场

1. 天线的交变电流：

$$
\boldsymbol{j}(\boldsymbol{r}',t') = \boldsymbol{j}(\boldsymbol{r}')\mathrm{e}^{-\mathrm{i}\omega t'} \tag{8.7}
$$

2. 天线产生的磁矢势：

$$
\boldsymbol{A}(\boldsymbol{r},t) = \boldsymbol{A}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\omega t} = \left(\frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{j}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\right)\mathrm{e}^{-\mathrm{i}\omega t} \tag{8.8}
$$

3. 天线产生的电磁场：

$$
\boldsymbol{B}(\boldsymbol{r},t) = (\nabla\times\boldsymbol{A}(\boldsymbol{r}))\mathrm{e}^{-\mathrm{i}\omega t},\quad \boldsymbol{E}(\boldsymbol{r},t) = \frac{\mathrm{i}c}{k}(\nabla\times\boldsymbol{B}(\boldsymbol{r}))\mathrm{e}^{-\mathrm{i}\omega t} \tag{8.9}
$$

8.2.2 天线在远场产生的电磁场

1. 级数展开：

$$
|\boldsymbol{r}-\boldsymbol{r}'| = |\boldsymbol{r}|-\boldsymbol{r}'\cdot\boldsymbol{e}_r+\cdots \tag{8.10}
$$

$$
\boldsymbol{A}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi}\int_V \boldsymbol{j}(\boldsymbol{r}')\frac{\mathrm{e}^{\mathrm{i}k(r-\boldsymbol{r}'\cdot\boldsymbol{e}_r)}}{r}\mathrm{d}V' = \frac{\mu_0}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\int_V \boldsymbol{j}(\boldsymbol{r}')(1-\mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{e}_r)\mathrm{d}V' \tag{8.11}
$$

2. 电偶极辐射的磁矢势：

$$
\begin{aligned}
\boldsymbol{A}^{(1)}(\boldsymbol{r}) &= \frac{\mu_0\mathrm{e}^{\mathrm{i}kr}}{4\pi}\int_V \boldsymbol{j}(\boldsymbol{r}')\mathrm{d}V' = \frac{\mu_0\mathrm{e}^{\mathrm{i}kr}}{4\pi}\int_V (-\boldsymbol{r}'(\nabla_{r'}\cdot\boldsymbol{j}(\boldsymbol{r}')))\mathrm{d}V' \\
&= -\frac{\mu_0}{4\pi}\mathrm{e}^{\mathrm{i}kr}\int_V \mathrm{i}\omega\rho(\boldsymbol{r}')\boldsymbol{r}'\mathrm{d}V' = -\frac{\mathrm{i}\mu_0\omega}{4\pi r}\mathrm{e}^{\mathrm{i}kr}\boldsymbol{p} = \frac{\mu_0\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\dot{\boldsymbol{p}}
\end{aligned} \tag{8.12}
$$

3. 电偶极辐射的电磁场：

$$
\boldsymbol{B}^{(1)}(\boldsymbol{r}) = \nabla\times\boldsymbol{A}^{(1)}(\boldsymbol{r}) = -\frac{\mathrm{i}\mu_0\omega}{4\pi}\frac{\mathrm{e}^{-\mathrm{i}kr}}{r}\left(\mathrm{i}k-\frac{1}{r}\right)\boldsymbol{e}_r\times\boldsymbol{p} \approx \frac{\mu_0\omega k}{4\pi}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\boldsymbol{e}_r\times\boldsymbol{p} = \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi\varepsilon_0 c^3 r}\ddot{\boldsymbol{p}}\times\boldsymbol{e}_r \tag{8.13}
$$

$$
\boldsymbol{E}^{(1)}(\boldsymbol{r}) = \frac{\mathrm{i}c}{k}\nabla\times\boldsymbol{B}^{(1)}(\boldsymbol{r}) \approx -\frac{\omega^2\mathrm{e}^{\mathrm{i}kr}}{4\pi\varepsilon_0 c^2 r}\boldsymbol{e}_r\times(\boldsymbol{e}_r\times\boldsymbol{p}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi\varepsilon_0 c^2 r}(\ddot{\boldsymbol{p}}\times\boldsymbol{e}_r)\times\boldsymbol{e}_r \tag{8.14}
$$

球坐标系下的电磁场：

$$
\boldsymbol{B}^{(1)}(\boldsymbol{r}) = -\frac{\omega^2\mathrm{e}^{\mathrm{i}kr}}{4\pi\varepsilon_0 c^3 r}|\boldsymbol{p}|\sin\theta\boldsymbol{e}_\varphi,\quad \boldsymbol{E}^{(1)} \approx -\frac{\omega^2\mathrm{e}^{\mathrm{i}kr}}{4\pi\varepsilon_0 c^2 r}|\boldsymbol{p}|\sin\theta\boldsymbol{e}_\theta \tag{8.15}
$$

4. 电偶极辐射的平均能流密度：

$$\left\langle \boldsymbol{S}^{(1)}(\boldsymbol{r})\right\rangle = \frac{1}{T}\int_0^T \frac{1}{\mu_0}\mathrm{Re}\boldsymbol{E}^{(1)}(\boldsymbol{r},t)\times\mathrm{Re}\boldsymbol{B}^{(1)}(\boldsymbol{r},t) = \frac{\omega^4|\boldsymbol{p}|^2}{32\pi^2\varepsilon_0 c^3 r^2}\sin^2\theta\boldsymbol{e}_r \tag{8.16}$$

5. 电偶极辐射的辐射功率：

$$P^{(1)} = \int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi} r^2\sin\theta\mathrm{d}\theta\left\langle \boldsymbol{S}^{(1)}(\boldsymbol{r})\right\rangle\cdot\boldsymbol{e}_r = \frac{\omega^4|\boldsymbol{p}|^4}{16\pi\varepsilon_0 c^3}\int_0^{\pi}\sin^3\theta\mathrm{d}\theta = \frac{\omega^4|\boldsymbol{p}|^4}{12\pi\varepsilon_0 c^3} \tag{8.17}$$

9 带电粒子的辐射

9.1 Lienard-Wiechert 势

1. 静止系中的电磁势：

$$\widetilde{\boldsymbol{A}}(\widetilde{\boldsymbol{r}}) = 0, \quad \widetilde{\varphi}(\widetilde{\boldsymbol{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\widetilde{\boldsymbol{r}}-\widetilde{\boldsymbol{r}}'|} \tag{9.1}$$

2. Lienard-Wiechert 势：

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{q\boldsymbol{v}'(t')}{4\pi\varepsilon_0 c^2\left(|\boldsymbol{r}-\boldsymbol{r}'(t')| - \frac{\boldsymbol{v}'(t')}{c}\cdot(\boldsymbol{r}-\boldsymbol{r}'(t'))\right)} \tag{9.2}$$

$$\varphi(\boldsymbol{r},t) = \frac{q}{4\pi\varepsilon_0\left(|\boldsymbol{r}-\boldsymbol{r}'(t')| - \frac{\boldsymbol{v}'(t')}{c}\cdot(\boldsymbol{r}-\boldsymbol{r}'(t'))\right)} \tag{9.3}$$

10 媒介质对电磁波的影响

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