习题4 答 案

1． 1）正确　2）错误　3）错误　4）正确

2． 验证略

3． $x^{'}=Ax$，　$A=\begin{pmatrix}16 & 14 & 38\\ -9 & -7 & -18\\ -4 & -4 & -11\end{pmatrix}$

4． 证明：设 x_i 是基本解组的第 i 个列向量，所以，$x_i^{'}=A_1(t)x_i=A_2(t)x_i$，

$(A_1(t)-A_2(t))x_i=0$，$(A_1(t)-A_2(t))\begin{pmatrix}x_1 & x_2 & \cdots & x_n\end{pmatrix}=0$。

$\Phi(t)=\begin{pmatrix}x_1(t) & x_2(t) & \cdots & x_n(t)\end{pmatrix}$ 是基本解组，所以 $A_1(t)=A_2(t)$

5．（1）对于满足微分系统的解 $x(t)$，都可以表示为 $x(t)=\phi(t)c$，

则 $x(t+kT)=\phi(t+kT)c$，

由于 $x^{'}=A(t)x$，所以 $x^{'}(t+kT)=A(t)x(t+kT)=A(t+kT)x(t+kT)$，

即 $x(t+kT)$ 也是微分方程组的解。

所以 $\phi^{'}(t+kT)=A(t)\phi(t+kT)=A(t+kT)\phi(t+kT)$

即证得 $\phi(t+kT)$ 也是微分方程组的基本解组

（2）当k=1时，由推论4.5可知，存在非奇异方阵B，使 $\Phi(t+T)=\Phi(t)B$, 同理

$\Phi(t+2T)=\Phi(t+T)B=\Phi(t)BB=\Phi(t)B^2$，……$\Phi(t+kT)=\Phi(t)B^k$。

6． 利用常数变易法或消元法

$$x(t)=c_1\begin{pmatrix}t^2\\ -t\end{pmatrix}+c_2\begin{pmatrix}t^2\ln(t)\\ -t-t\ln(t)\end{pmatrix}+\begin{pmatrix}\dfrac{t^4}{4}-\dfrac{t^2}{2}\ln^2(t)\\ -\dfrac{3t^3}{4}+\dfrac{t}{2}\ln^2(t)+t\ln(t)+t\end{pmatrix}$$

7．

$$x_1(t)=\frac{1}{2}(2c_1-2c_2t+c_3t^2)e^{-t},\ x_2(t)=(c_2-c_3t)e^{-t},x_3(t)=c_3e^{-t}$$

8．

$$x(t)=2(t-1)e^t,\ y(t)=3(t-1)e^t,z(t)=-te^t$$

9．提示：从第一个方程解出y，代入到第二个方程，即变为二阶常系数线性方程，可以用欧拉法解。

$$\frac{d^2x}{dt^2}-(a_{11}+a_{22})\frac{dx}{dt}+(a_{11}a_{22}-a_{12}a_{21})x=0.$$

$$x=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}，\ y(t)=\frac{1}{a_{12}}\left(c_1(\lambda_1-a_{11})e^{\lambda_1 t}+c_2(\lambda_2-a_{22})e^{\lambda_2 t}\right)$$

$$\lambda_1,\lambda_2=\frac{1}{2}\left(a_{11}+a_{22}\pm\sqrt{\left(a_{11}+a_{22}\right)^2-4\left(a_{11}a_{22}-a_{12}a_{21}\right)}\right)$$

10．

（1）
$$x(t)=c_1e^t+c_2e^{-2t}+c_3e^{-t}$$
$$y(t)=2c_1e^t+\frac{c_2}{2}e^{-2t}$$

(2）
$$x(t)=c_1e^{-t}+c_2te^{-t}$$
$$y(t)=c_2e^{-t}$$

11.

1）$\begin{cases}x(t)=c_1e^t+c_2e^{-t}\\ y(t)=c_1e^t+2c_2e^{-t}\end{cases}$，

2）$\begin{cases}x(t)=e^{2t}(c_1+c_2t)\\ y(t)=-e^{2t}(c_1+c_2+c_2t)\end{cases}$

3）$\begin{cases}x(t)=e^t(c_1+c_2t)\\ y(t)=e^t(2c_1+2c_2t+c_2)\\ z(t)=-c_1e^t-c_2(t+1)e^t+c_3e^{2t}\end{cases}$

4)

$$x=c_1e^t+c_2e^{3t}+c_3e^{2t}$$
$$y=\frac{2}{3}c_1e^t+c_2e^{3t}+\frac{1}{4}c_3e^{2t}$$
$$z=c_1e^t+\frac{3}{2}c_2e^{3t}+\frac{1}{4}c_3e^{2t}$$

5)

$$x = e^t\left(c_1 + c_2 \sin\sqrt{3}t + c_3 \cos\sqrt{3}t\right)$$

$$y = \frac{1}{2}e^t\left(2c_1 - c_2 \sin\sqrt{3}t + c_2\sqrt{3}\cos\sqrt{3}t - c_3 \cos\sqrt{3}t - c_3\sqrt{3}\sin\sqrt{3}t\right)$$

$$z = \frac{1}{2}e^t\left(2c_1 - c_2 \sin\sqrt{3}t - c_2\sqrt{3}\cos\sqrt{3}t - c_3 \cos\sqrt{3}t + c_3\sqrt{3}\sin\sqrt{3}t\right)$$

6) $$\begin{cases} x(t) = \frac{1}{2}e^{-t}(2c_1 - 2c_2 t + c_3 t^2) \\ y(t) = e^{-t}(c_2 - c_3 t) \\ z(t) = c_3 e^{-t} \end{cases}$$

12.

1)

```
restart:with(DEtools):
eq1:=diff(x1(t),t)=3*x1(t)+2*x2(t)+2*x3(t);
eq2:=diff(x2(t),t)=x1(t)+4*x2(t)+x3(t);
eq3:=diff(x3(t),t)=-2*x1(t)-4*x2(t)-x3(t);
dsolve([eq1,eq2,eq3],{x1(t),x2(t),x3(t)});
```

$$eq1 := \frac{\partial}{\partial t}\mathrm{x1}(t) = 3\,\mathrm{x1}(t) + 2\,\mathrm{x2}(t) + 2\,\mathrm{x3}(t)$$

$$eq2 := \frac{\partial}{\partial t}\mathrm{x2}(t) = \mathrm{x1}(t) + 4\,\mathrm{x2}(t) + \mathrm{x3}(t)$$

$$eq3 := \frac{\partial}{\partial t}\mathrm{x3}(t) = -2\,\mathrm{x1}(t) - 4\,\mathrm{x2}(t) - \mathrm{x3}(t)$$

$$\{\mathrm{x3}(t) = -\_C2\,\mathbf{e}^t - \_C1\,\mathbf{e}^{(3\,t)},\ \mathrm{x1}(t) = \_C2\,\mathbf{e}^t + \_C3\,\mathbf{e}^{(2\,t)},$$
$$\mathrm{x2}(t) = -\frac{1}{2}\_C3\,\mathbf{e}^{(2\,t)} + \_C1\,\mathbf{e}^{(3\,t)}\}$$

2)

```
restart:with(DEtools):
eq1:=diff(y1(t),t)=2*y1(t)+y2(t);
eq2:=diff(y2(t),t)=-y1(t)+y3(t);
eq3:=diff(y3(t),t)=y1(t)+3*y2(t)+y3(t);
dsolve([eq1,eq2,eq3],{y1(t),y2(t),y3(t)});
```

$$eq1 := \frac{\partial}{\partial t}\mathrm{y1}(t) = 2\,\mathrm{y1}(t) + \mathrm{y2}(t)$$

$$eq2 := \frac{\partial}{\partial t}\mathrm{y2}(t) = -\mathrm{y1}(t) + \mathrm{y3}(t)$$

$$eq3 := \frac{\partial}{\partial t}\mathrm{y3}(t) = \mathrm{y1}(t) + 3\,\mathrm{y2}(t) + \mathrm{y3}(t)$$

$$\{ y1(t) = \_C1\, e^{(-t)} + \_C2\, e^{(2t)} + \_C3\, e^{(2t)}\, t,$$
$$y3(t) = 4\,\_C1\, e^{(-t)} + \_C2\, e^{(2t)} + \_C3\, e^{(2t)}\, t + 2\,\_C3\, e^{(2t)},$$
$$y2(t) = -3\,\_C1\, e^{(-t)} + \_C3\, e^{(2t)} \}$$

3)

```
> restart:with(DEtools):
eq1:=diff(x1(t),t)=2*x1(t)+x2(t);
eq2:=diff(x2(t),t)=-x1(t)+x3(t);
eq3:=diff(x3(t),t)=x1(t)+3*x2(t)+x3(t);
dsolve([eq1,eq2,eq3,x1(0)=1,x2(0)=1,x3(0)=1],
{x1(t),x2(t),x3(t)});
```

$$eq1 := \frac{\partial}{\partial t} x1(t) = 2\, x1(t) + x2(t)$$

$$eq2 := \frac{\partial}{\partial t} x2(t) = -x1(t) + x3(t)$$

$$eq3 := \frac{\partial}{\partial t} x3(t) = x1(t) + 3\, x2(t) + x3(t)$$

$$\{ x3(t) = -\frac{8}{9} e^{(-t)} + \frac{17}{9} e^{(2t)} + \frac{1}{3} e^{(2t)}\, t,\ x2(t) = \frac{2}{3} e^{(-t)} + \frac{1}{3} e^{(2t)},$$
$$x1(t) = -\frac{2}{9} e^{(-t)} + \frac{11}{9} e^{(2t)} + \frac{1}{3} e^{(2t)}\, t \}$$

4)

```
restart:with(DEtools):
eq1:=diff(y1(t),t)=y1(t)+y2(t)-3;
eq2:=diff(y2(t),t)=-2*y1(t)+3*y2(t)+1;
dsolve([eq1,eq2,y1(0)=0,y2(0)=0],{y1(t),y2(t)});
```

$$eq1 := \frac{\partial}{\partial t} y1(t) = y1(t) + y2(t) - 3$$

$$eq2 := \frac{\partial}{\partial t} y2(t) = -2\, y1(t) + 3\, y2(t) + 1$$

$$\{ y1(t) = 2 + e^{(2t)} (\sin(t) - 2\cos(t)),\ y2(t) = 1 - e^{(2t)} (-3\sin(t) + \cos(t)) \}$$

13. （1）

$$\frac{dx_1(t)}{dt} = f_0(t) - \frac{k_{12}+k_{13}}{v_1} x_1(t) + \frac{k_{21}}{v_2} x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{k_{12}}{v_1} x_1(t) - \frac{k_{21}}{v_2} x_2(t)$$

（2）

$$\frac{dx_1(t)}{dt} = -\frac{k_{12}+k_{13}}{v_1}x_1(t) + \frac{k_{21}}{v_2}x_2(t)$$
$$\frac{dx_2(t)}{dt} = \frac{k_{12}}{v_1}x_1(t) - \frac{k_{21}}{v_2}x_2(t)$$
$$x_1(0) = D_0, \quad x_2(0) = 0$$

（3）

$$\frac{dx_1(t)}{dt} = k_0 - \frac{k_{12}+k_{13}}{v_1}x_1(t) + \frac{k_{21}}{v_2}x_2(t)$$
$$\frac{dx_2(t)}{dt} = \frac{k_{12}}{v_1}x_1(t) - \frac{k_{21}}{v_2}x_2(t)$$
$$x_1(0) = 0, \quad x_2(0) = 0$$

14. 模型为

$$t \le 365$$
$$\frac{dx_1(t)}{dt} = -(a_{01}+a_{21}+a_{31})\,x_1(t) + a_{12}x_2(t) + a_{13}x_3(t) + b$$
$$\frac{dx_2(t)}{dt} = a_{21}x_1(t) - (a_{02}+a_{12})x_2(t)$$
$$\frac{dx_3(t)}{dt} = a_{31}x_1(t) - a_{13}x_3(t)$$
$$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = 0,$$

$$t \ge 365$$
$$\frac{dx_1(t)}{dt} = -(a_{01}+a_{21}+a_{31})\,x_1(t) + a_{12}x_2(t) + a_{13}x_3(t)$$
$$\frac{dx_2(t)}{dt} = a_{21}x_1(t) - (a_{02}+a_{12})x_2(t)$$
$$\frac{dx_3(t)}{dt} = a_{31}x_1(t) - a_{13}x_3(t)$$
$$x_1(0) = x_1(365), \quad x_2(0) = x_2(365), \quad x_3(0) = x_3(365)$$

下面利用Maple求解

时间分为两段: [0,365]和[365,1460]

#定义常数

a01:= 0.021; a02:=0.016; a12:=0.012; a13:=0.000035; a21:=0.011; a31:=0.0039; L:=49.3;

#定义矩阵和向量

A := matrix(3,3,[-(a01+a21+a31), a12, a13, a21, -(a02+a12), 0, a31, 0, -a13]);

b :=matrix(3,1,[L, 0, 0]);

#定义方程

equn1:=diff(x1(t),t)=-(a01+a21+a31)*x1(t)+a12*x2(t)+a13*x3(t)+L;
equn2:=diff(x2(t),t)=a21*x1(t)-(a02+a12)*x2(t);
equn3:=diff(x3(t),t)=a31*x1(t)-a13*x3(t);

#解初始值问题

dsolve({equn1,equn2,equn3,x1(0)=0,x2(0)=0,x3(0)=0},
{x1(t),x2(t),x3(t)});

结果太复杂,利用迭加原理、特征值和特征向量法

#非齐次方程的特解

with(linalg): with(plots):
xe := evalm(-(inverse(A)&*b));

#特征值和特征向量

eigenvals(A);
eigenvects(A);
lambda:=[-.3061796847e-4, -.1980315266e-1,-.4410122938e-1];
v1:=[.1124436946e-1, .442226656e-2, 10.00746814];
v2:=[-.5975248926, -.8018660787, .1178839056];
v3:=[-.8256380196, .5640574390, .7307156328e-1];

#特征向量组成矩阵

P:=augment(v1,v2,v3);

#得到解的表达式

x1:=c[1]*P[1,1]*exp(lambda[1]*t)+c[2]*P[1,2]*exp(lambda[2]*t)+c[3]*P[1,3]*exp(lambda[3]*t)+xe[1,1];
x2 :=
c[1]*P[2,1]*exp(lambda[1]*t)+c[2]*P[2,2]*exp(lambda[2]*t)+c[3]*P[2,3]*exp(lambda[3]*t)+xe[2,1];
x3 :=
c[1]*P[3,1]*exp(lambda[1]*t)+c[2]*P[3,2]*exp(lambda[2]*t)+c[3]*P[3,3]*exp(lambda[3]*t)+xe[3,1];

#在解的表达式用**t=0**代入

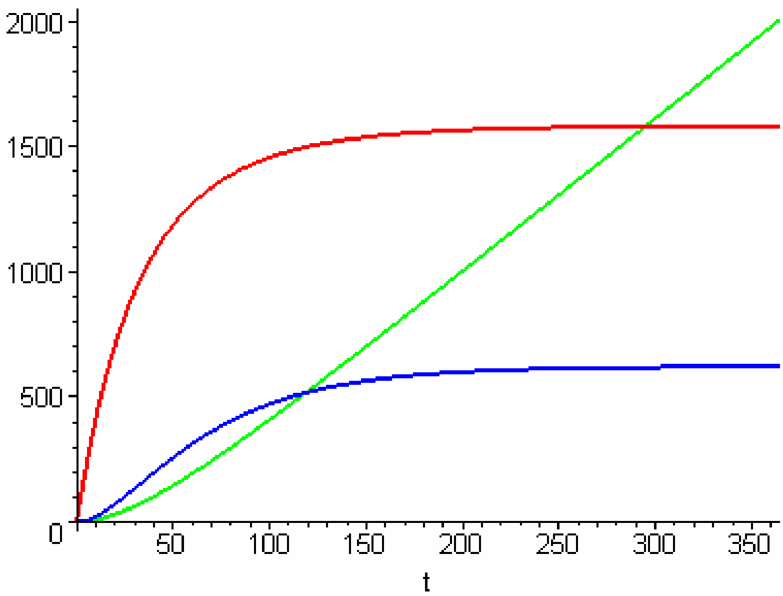
x10:=simplify(subs(t=0,x1));
x20:=simplify(subs(t=0,x2));
x30:=simplify(subs(t=0,x3));

#求解常数

solve({x10=0,x20=0,x30=0},{c[1],c[2],c[3]});
assign(%);

#作图

plot([x1, x2, x3], t=0..365, color=[red,blue,green], thickness=2); plot1:=%:




```
#重新显示解
x1;x2;x3;
#求极限
limit(x1,t=infinity);
limit(x2,t=infinity);
limit(x3,t=infinity);
#得到365天后解的表达式
xx1:=cc[1]*P[1,1]*exp(lambda[1]*t)+cc[2]*P[1,2]*exp(lambda[2]*t)+cc[3]*P[1,3]
*exp(lambda[3]*t);
xx2 :=
cc[1]*P[2,1]*exp(lambda[1]*t)+cc[2]*P[2,2]*exp(lambda[2]*t)+cc[3]*P[2,3]*exp(
lambda[3]*t);
xx3 :=
cc[1]*P[3,1]*exp(lambda[1]*t)+cc[2]*P[3,2]*exp(lambda[2]*t)+cc[3]*P[3,3]*exp(
lambda[3]*t);
#  计算解在t=365时的值
x1365:=simplify(subs(t=365,x1));
x2365:=simplify(subs(t=365,x2));
x3365:=simplify(subs(t=365,x3));
xx1365:=simplify(subs(t=365,xx1));
xx2365:=simplify(subs(t=365,xx2));
xx3365:=simplify(subs(t=365,xx3));
#求解常数
solve({xx1365=x1365,xx2365=x2365,xx3365=x3365},
                {cc[1],cc[2],cc[3]});
assign(%);
#重新显示解
xx1;xx2;xx3;
```

#求极限

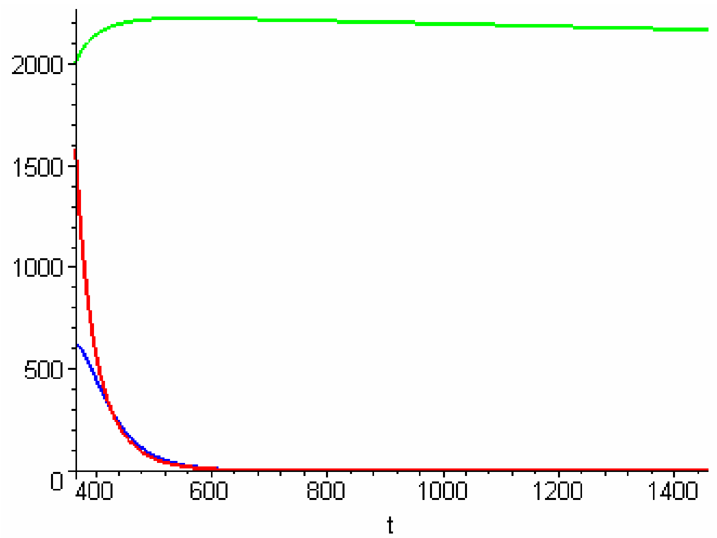
limit(xx1,t=infinity);

limit(xx2,t=infinity);

limit(xx3,t=infinity);

#作图

plot([xx1, xx2, xx3], t=365..1460, color=[red,blue,green], thickness=2); plot2:=%:



#将两个图画在一个图上

display(plot1,plot2);