# 答 案 4.4

**1.** (1) $x_1(t)=(2e^{10t},e^{10t})^T$, $x_2(t)=(-3e^{3t},2e^{3t})^T$

(2) $x_1(t)=(-e^{t},e^{t})^T$, $x_2(t)=(2e^{4t},e^{4t})^T$

(3) $x_1(t)=(e^{-t},e^{-t})^T$, $x_2(t)=(4e^{2t},e^{2t})^T$

(4) $x_1(t)=(-e^{2t},2e^{2t})^T$, $x_2(t)=(-e^{3t},e^{3t})^T$

(5) $x_1(t)=(e^{t},e^{t})^T$, $x_2(t)=(e^{t}+te^{t},\frac{3}{2}e^{t}+te^{t})^T$

(6) $x_1(t)=(e^{2t},e^{2t})^T$, $x_2(t)=(-e^{-2t},3e^{-2t})^T$

(7) $x_1(t)=(1,2)^T$, $x_2(t)=(-e^{3t},e^{3t})^T$

(8) $x_1(t)=(e^{t},e^{t})^T$, $x_2(t)=(2e^{t}+te^{t},e^{t}+te^{t})^T$

(9) $x_1(t)=(\cos 2t,\cos 2t+2\sin 2t)^T$, $x_2(t)=(\sin 2t,\sin 2t-2\cos 2t)^T$,

(10) $x_1(t)=(e^{2t}\cos t,-e^{2t}\sin t)^T$, $x_2(t)=(e^{2t}\sin t,e^{2t}\cos t)^T$

**2.**

(1) $$\phi(t)=\begin{bmatrix} 4 & e^{-3t} & (2+t)e^{-3t} \\ 4 & -2e^{-3t} & (-3-2t)e^{-3t} \\ 1 & e^{-3t} & (1+t)e^{-3t} \end{bmatrix}$$

(2) $$\phi(t)=\begin{bmatrix} e^{t} & -e^{-2t} & -e^{-2t} \\ e^{t} & e^{-2t} & 0 \\ e^{t} & 0 & e^{-2t} \end{bmatrix}$$

(3) $$\phi(t)=e^{-t}\begin{bmatrix} 1 & -t & \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}$$

(4) $$\phi(t)=\begin{bmatrix} 0 & e^{4t} & e^{t} \\ e^{4t} & 0 & 0 \\ 0 & 2e^{4t} & e^{t} \end{bmatrix}$$

(5) $\phi(t)=\begin{bmatrix} e^{-t} & e^{2t} & 0 \\ e^{-t} & e^{2t} & e^{-2t} \\ 0 & e^{2t} & e^{-2t} \end{bmatrix}$

（6） $\phi(t)=\begin{pmatrix} 2e^{10t} & -3e^{3t} & 0 & 0 \\ e^{10t} & 2e^{3t} & 0 & 0 \\ 0 & 0 & 2e^{10t} & -3e^{3t} \\ 0 & 0 & e^{10t} & 2e^{3t} \end{pmatrix}$

**3**．证明：(1) $\exp(c_1A)\cdot\exp(x_2A)=(E+c_1A+\frac{c_1^2A^2}{2!}+\cdots)(E+c_2A+\frac{c_2^2A^2}{2!}+\cdots)$

$$=E+(c_1A+c_2A)+\frac{1}{2!}(c_1^2A^2+2c_1Ac_2A+c_2^2A^2)+\cdots$$

由二项式定理及 $A\cdot A^2=A^2\cdot A,$

得 $\exp(c_1A+c_2A)=E+(c_1A+c_2A)+\frac{1}{2!}(c_1^2A^2+2c_1Ac_2A+c_2^2A^2)+\cdots$

比较二式得 $\exp(c_1A+c_2A)=\exp(c_1A)\cdot\exp(c_2A)$

(2) 由(1) $e^{\overbrace{(A+A+\cdots+A)}^{k个}}=e^{kA}$

$e^{\overbrace{(A+A+\cdots+A)}^{k个}}=e^A\cdot e^A\cdot e^A\cdot\cdots\cdot e^A=(e^A)^k$

$\therefore(\exp A)^k=\exp kA$

**4**．解： $\exp At=E+At+\frac{A^2t^2}{2!}+\cdots+\frac{A^kt^k}{k!}+\cdots=\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}$

$A^2=\alpha A$， $A^3=A\cdot A^2=\alpha A^2=\alpha^2A$

$A^4=A\cdot A^3=\alpha^3A,\cdots\cdots A^k=\alpha^{k-1}A$

$$\exp At=E+At+\frac{\alpha A}{2!}t^2+\cdots+\frac{\alpha^{k-1}A}{k!}t^k+\cdots=E+\sum_{k=1}^{\infty}\frac{\alpha^{k-1}A}{k!}t^k=E+\left(e^{\alpha t}-1\right)\frac{A}{\alpha}$$

**5**．（1）验证

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restart:with(linalg):
A:=matrix(5,5,[1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1]);
B:=matrix(5,5,[-4,1,1,1,1,1,-4,1,1,1,1,1,
-4,1,1,1,1,1,-4,1,1,1,1,1,-4]);
multiply(A,B);
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$B := \begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 1 & -4 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

（2）利用上一题目的结果知道： $\exp At = E + \dfrac{e^{5t}-1}{5} A$

**6.** （1）$\Phi(t)=\begin{bmatrix} \cos\sqrt{3}t & \sin\sqrt{3}t \\ \frac{1}{2}\cos\sqrt{3}t - \frac{\sqrt{3}}{2}\sin\sqrt{3}t & \frac{1}{2}\sin\sqrt{3}t + \frac{\sqrt{3}}{2}\cos\sqrt{3}t \end{bmatrix}$

$$\exp At = \begin{bmatrix} \cos\sqrt{3}t - \frac{1}{2}\sin\sqrt{3}t & \frac{2}{\sqrt{3}}\sin\sqrt{3}t \\ \frac{2-\sqrt{3}}{4}\cos\sqrt{3}t - \frac{1+2\sqrt{3}t}{4}\sin\sqrt{3}t & \frac{1}{\sqrt{3}}\sin\sqrt{3}t + \cos\sqrt{3}t \end{bmatrix}$$

(2) $\Phi(t)=\begin{bmatrix} e^{-t} & e^{5t} \\ -e^{-t} & 2e^{5t} \end{bmatrix}$

$$\exp At = \begin{bmatrix} \frac{2}{3}e^{-t} + \frac{1}{3}e^{5t} & -\frac{1}{3}e^{-t} + \frac{1}{3}e^{5t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{5t} & \frac{1}{3}e^{-t} + \frac{2}{3}e^{5t} \end{bmatrix}$$

**7.**

**（1）**

```
restart:
a11:=1: a12:=-2: a21:=1: a22:=-1: x0:=0: y0:=-1:
eq1:=diff(x(t),t)=a11*x(t)+a12*y(t);
eq2:=diff(y(t),t)=a21*x(t)+a22*y(t);
sol:=dsolve({eq1,eq2,x(0)=x0,y(0)=y0},{x(t),y(t)});
```

$$eq1 := \frac{\partial}{\partial t} x(t) = x(t) - 2\, y(t)$$

$$eq2 := \frac{\partial}{\partial t} y(t) = x(t) - y(t)$$

$$sol := \{ y(t) = -\cos(t) + \sin(t), x(t) = 2 \sin(t) \}$$

（2）

```
restart:
a11:=-6: a12:=-1: a21:=1: a22:=-4: x0:=0: y0:=-1:
eq1:=diff(x(t),t)=a11*x(t)+a12*y(t);
eq2:=diff(y(t),t)=a21*x(t)+a22*y(t);
sol:=dsolve({eq1,eq2,x(0)=x0,y(0)=y0},{x(t),y(t)});
```

$$eq1 := \frac{\partial}{\partial t} x(t) = -6\, x(t) - y(t)$$

$$eq2 := \frac{\partial}{\partial t} y(t) = x(t) - 4\, y(t)$$

$$sol := \{ y(t) = -\mathbf{e}^{(-5\, t)} (1 + t), x(t) = \mathbf{e}^{(-5\, t)}\, t \}$$

（3）

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restart:
a11:=2: a12:=0: a21:=1: a22:=1: x0:=1: y0:=2:
eq1:=diff(x(t),t)=a11*x(t)+a12*y(t);
eq2:=diff(y(t),t)=a21*x(t)+a22*y(t);
sol:=dsolve({eq1,eq2,x(0)=x0,y(0)=y0},{x(t),y(t)});
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$$eq1 := \frac{\partial}{\partial t} x(t) = 2\, x(t)$$

$$eq2 := \frac{\partial}{\partial t} y(t) = x(t) + y(t)$$

$$sol := \{ x(t) = \mathbf{e}^{(2\, t)}, y(t) = \mathbf{e}^{(2\, t)} + \mathbf{e}^{t} \}$$

（4）

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restart:
a11:=0: a12:=2: a21:=-4: a22:=-4: x0:=2: y0:=-1:
eq1:=diff(x(t),t)=a11*x(t)+a12*y(t);
eq2:=diff(y(t),t)=a21*x(t)+a22*y(t);
sol:=dsolve({eq1,eq2,x(0)=x0,y(0)=y0},{x(t),y(t)});
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$$eq1 := \frac{\partial}{\partial t} x(t) = 2\, y(t)$$

$$eq2 := \frac{\partial}{\partial t} y(t) = -4\, x(t) - 4\, y(t)$$

$$sol := \{ y(t) = \mathbf{e}^{(-2\, t)} (-3 \sin(2\, t) - \cos(2\, t)), x(t) = \mathbf{e}^{(-2\, t)} (\sin(2\, t) + 2 \cos(2\, t)) \}$$