答案 5.4

1.

利用极坐标　$x = r\cos\theta, y = r\sin\theta,$　条件变为

$$\lim_{r\to 0}\frac{\varphi(r\cos\theta, r\sin\theta)}{r} = 0,\quad \lim_{r\to 0}\frac{\psi(r\cos\theta, r\sin\theta)}{r} = 0$$

1）解:　$$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{r^2\cos\theta\sin\theta}{r} = 0$$

此几乎线性系统可等效为线性系统

奇点类型是：稳定焦点

2）$\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0$，奇点类型：不稳定结点

3）同 1)，2)　$\lim_{(x,y)>(0,0)}\frac{x(x^2+y^2)}{\sqrt{x^2+y^2}} = 0$　　$\lim_{(x,y)\to(0,0)}\frac{y(x^2+y^2)}{\sqrt{x^2+y^2}} = 0$

奇点类型：稳定焦点

4）用极坐标代换　$x = r\cos\theta,\quad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)}\frac{2x^2+y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{2r^2\cos^2\theta - r^2\sin^2\theta}{r} = 0$$

$$\lim_{(x,y)\to(0,0)}\frac{x^3}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{r^3\cos^3\theta}{r} = 0$$

奇点类型：鞍点

5）用极坐标代换　$x = r\cos\theta,\quad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)}\frac{\mu x^2 y}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{\mu r^3\cos^2\theta\sin^2\theta}{r} = 0$$

$$p<0,\quad q>0,\quad \Delta = \mu^2 - 4\begin{cases} >0, & \text{不稳定结点} \\ =0, & \text{不稳定退化结点} \\ <0, & \text{不稳定焦点}\end{cases}$$

```
restart:with(DEtools):  a:=0.001;
ODES:=[diff(x(t),t)=y(t),
diff(y(t),t)=-x(t)+2*y(t)*(1-x(t)^2)]:
DEplot(ODES,[x(t),y(t)], t=-10..10,
[[x(0)=-8*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-6*a],
[x(0)=-8*a,y(0)=-4*a],[x(0)=-8*a,y(0)=-2*a],
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[x(0)=-8*a,y(0)=0],[x(0)=-8*a,y(0)=2*a],
[x(0)=-8*a,y(0)=4*a],[x(0)=-8*a,y(0)=6*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=8*a,y(0)=-6*a],[x(0)=8*a,y(0)=-4*a],
[x(0)=8*a,y(0)=-2*a],[x(0)=8*a,y(0)=0],
[x(0)=8*a,y(0)=2*a],[x(0)=8*a,y(0)=4*a],
[x(0)=8*a,y(0)=6*a],[x(0)=8*a,y(0)=8*a],
[x(0)=-6*a,y(0)=-8*a],[x(0)=-4*a,y(0)=-8*a],
[x(0)=-2*a,y(0)=-8*a],[x(0)=0,y(0)=-8*a],
[x(0)=2*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=6*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-6*a,y(0)=8*a],[x(0)=-4*a,y(0)=8*a],
[x(0)=-2*a,y(0)=8*a],[x(0)=0,y(0)=8*a],
[x(0)=2*a,y(0)=8*a],[x(0)=4*a,y(0)=8*a],
[x(0)=6*a,y(0)=8*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```
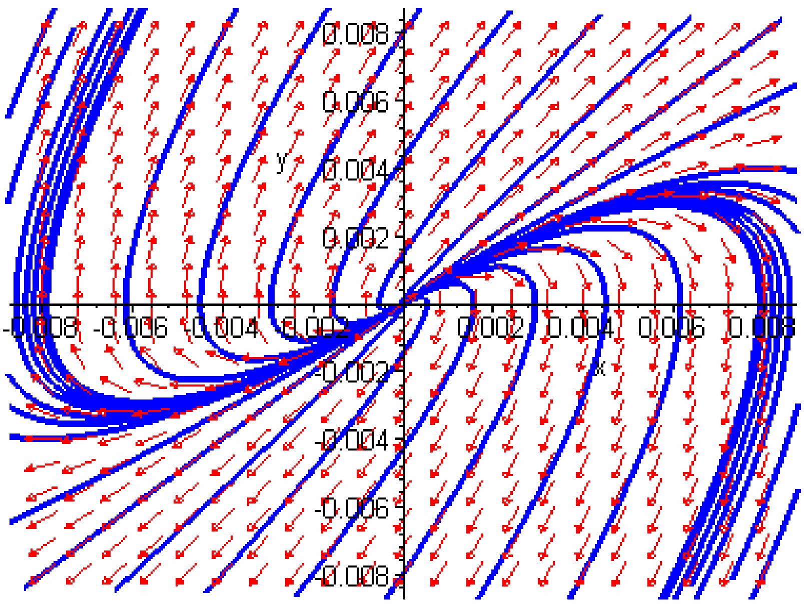

$$a := .001$$



6）由等价无穷小的公式　$1-e^{-x} \sim x,\quad \sin x \sim x$

奇点类型：不稳定焦点

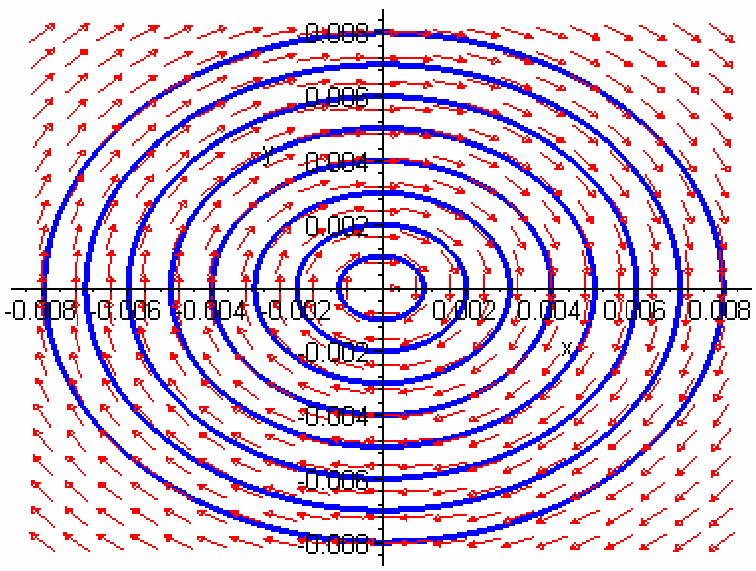
7）同 6）$1-\cos y \sim \dfrac{y^2}{2},\quad \sin y \sim y$

奇点类型：中心点

方程组有积分 $\cos y=(x+1)\left(\dfrac{2}{x+1}+\ln(x+1)+C\right)$

```
> restart:with(DEtools):  a:=0.001;
ODES:=[diff(x(t),t)=(1+x(t))*sin(y(t)),
diff(y(t),t)=1-x(t)-cos(y(t))]:
DEplot(ODES,[x(t),y(t)], t=-10..10,
[[x(0)=0,y(0)=a],[x(0)=0,y(0)=2*a],
[x(0)=0,y(0)=3*a],[x(0)=0,y(0)=4*a],
[x(0)=0,y(0)=5*a],[x(0)=0,y(0)=6*a],
[x(0)=0,y(0)=7*a],[x(0)=0,y(0)=8*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```

$$a := .001$$



8）由等价无穷小 $\sqrt{4+4y}-2\sim y+2,\quad e^{x+y}-1\sim x+y$

$$\sin ax\sim xa,\qquad \ln(1-4y)\sim -4y$$

$$\frac{dx}{dt}=-2x-y+\varphi(x,y),\quad \frac{dy}{dt}=ax-4y+\psi(x,y),$$

$$p=6>0,\quad q=8+a,\quad \Delta=4(1-a),\quad \begin{cases} a<-8, & \text{鞍点} \\ -8<a<1, & \text{稳定结点} \\ a>1, & \text{稳定焦点} \end{cases}$$

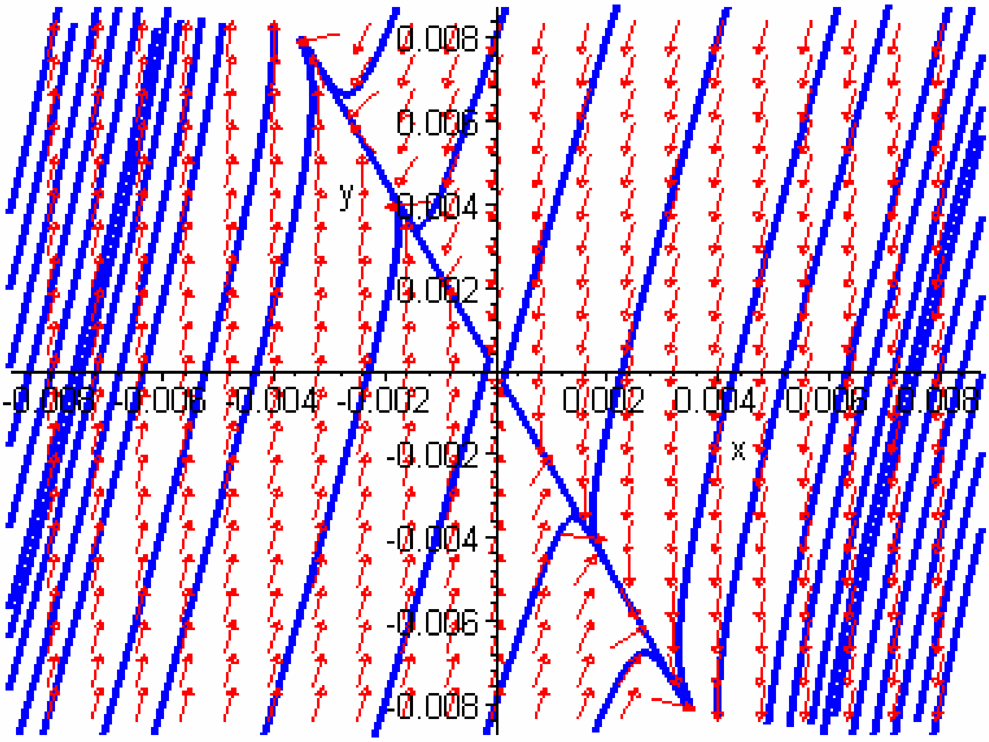
a=-10

```
restart:with(DEtools):  a:=0.001:aa:=-10:
DE931:=[diff(x(t),t)=sqrt(4+4*y(t))-2*exp(x(t)+y(t)),
diff(y(t),t)=sin(aa*x(t))+ln(1-4*y(t))];
```

```
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=-8*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-6*a],
[x(0)=-8*a,y(0)=-4*a],[x(0)=-8*a,y(0)=-2*a],
[x(0)=-8*a,y(0)=0],[x(0)=-8*a,y(0)=2*a],
[x(0)=-8*a,y(0)=4*a],[x(0)=-8*a,y(0)=6*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=8*a,y(0)=-6*a],[x(0)=8*a,y(0)=-4*a],
[x(0)=8*a,y(0)=-2*a],[x(0)=8*a,y(0)=0],
[x(0)=8*a,y(0)=2*a],[x(0)=8*a,y(0)=4*a],
[x(0)=8*a,y(0)=6*a],[x(0)=8*a,y(0)=8*a],
[x(0)=-6*a,y(0)=-8*a],[x(0)=-4*a,y(0)=-8*a],
[x(0)=-2*a,y(0)=-8*a],[x(0)=0,y(0)=-8*a],
[x(0)=2*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=6*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-6*a,y(0)=8*a],[x(0)=-4*a,y(0)=8*a],
[x(0)=-2*a,y(0)=8*a],[x(0)=0,y(0)=8*a],
[x(0)=2*a,y(0)=8*a],[x(0)=4*a,y(0)=8*a],
[x(0)=6*a,y(0)=8*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```

$DE931 :=$

$$\left[\frac{\partial}{\partial t}\mathrm{x}(t) = 2\sqrt{1+\mathrm{y}(t)} - 2\,\mathrm{e}^{(\mathrm{x}(t)+\mathrm{y}(t))}, \frac{\partial}{\partial t}\mathrm{y}(t) = -\sin(10\,\mathrm{x}(t)) + \ln(1-4\,\mathrm{y}(t))\right]$$



a=-8

```
restart:with(DEtools):  a:=0.001:aa:=-8:
DE931:=[diff(x(t),t)=sqrt(4+4*y(t))-2*exp(x(t)+y(t)),
diff(y(t),t)=sin(aa*x(t))+ln(1-4*y(t))];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=-8*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-6*a],
[x(0)=-8*a,y(0)=-4*a],[x(0)=-8*a,y(0)=-2*a],
[x(0)=-8*a,y(0)=0],[x(0)=-8*a,y(0)=2*a],
[x(0)=-8*a,y(0)=4*a],[x(0)=-8*a,y(0)=6*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=8*a,y(0)=-6*a],[x(0)=8*a,y(0)=-4*a],
[x(0)=8*a,y(0)=-2*a],[x(0)=8*a,y(0)=0],
[x(0)=8*a,y(0)=2*a],[x(0)=8*a,y(0)=4*a],
[x(0)=8*a,y(0)=6*a],[x(0)=8*a,y(0)=8*a],
[x(0)=-6*a,y(0)=-8*a],[x(0)=-4*a,y(0)=-8*a],
[x(0)=-2*a,y(0)=-8*a],[x(0)=0,y(0)=-8*a],
[x(0)=2*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=6*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-6*a,y(0)=8*a],[x(0)=-4*a,y(0)=8*a],
[x(0)=-2*a,y(0)=8*a],[x(0)=0,y(0)=8*a],
[x(0)=2*a,y(0)=8*a],[x(0)=4*a,y(0)=8*a],
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x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```
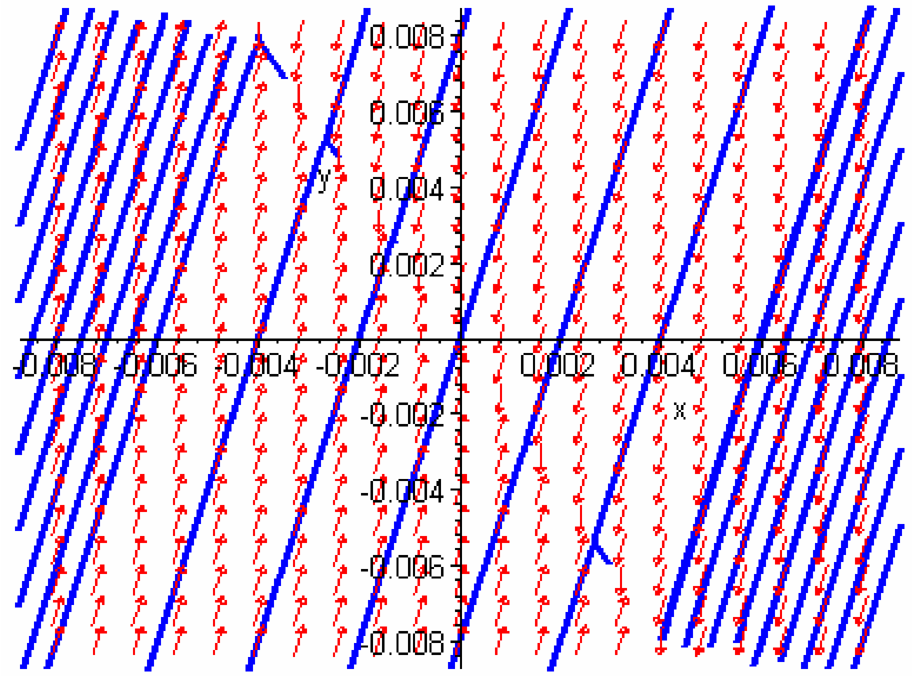

$$DE931 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = 2\sqrt{1+\mathrm{y}(t)} - 2\,\mathbf{e}^{(\mathrm{x}(t)+\mathrm{y}(t))}, \frac{\partial}{\partial t}\mathrm{y}(t) = -\sin(8\,\mathrm{x}(t)) + \ln(1-4\,\mathrm{y}(t))\right]$$



a=0

```
restart:with(DEtools):  a:=0.001:aa:=0:
DE931:=[diff(x(t),t)=sqrt(4+4*y(t))-2*exp(x(t)+y(t)),
diff(y(t),t)=sin(aa*x(t))+ln(1-4*y(t))];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=-8*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-6*a],
[x(0)=-8*a,y(0)=-4*a],[x(0)=-8*a,y(0)=-2*a],
[x(0)=-8*a,y(0)=0],[x(0)=-8*a,y(0)=2*a],
[x(0)=-8*a,y(0)=4*a],[x(0)=-8*a,y(0)=6*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=8*a,y(0)=-6*a],[x(0)=8*a,y(0)=-4*a],
[x(0)=8*a,y(0)=-2*a],[x(0)=8*a,y(0)=0],
[x(0)=8*a,y(0)=2*a],[x(0)=8*a,y(0)=4*a],
[x(0)=8*a,y(0)=6*a],[x(0)=8*a,y(0)=8*a],
[x(0)=-6*a,y(0)=-8*a],[x(0)=-4*a,y(0)=-8*a],
[x(0)=-2*a,y(0)=-8*a],[x(0)=0,y(0)=-8*a],
[x(0)=2*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=6*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-6*a,y(0)=8*a],[x(0)=-4*a,y(0)=8*a],
[x(0)=-2*a,y(0)=8*a],[x(0)=0,y(0)=8*a],
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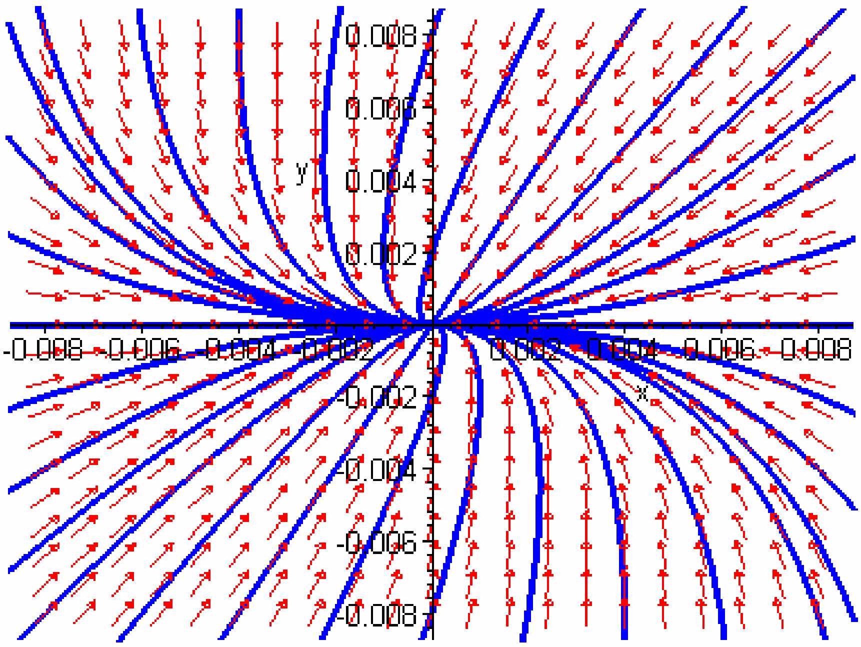
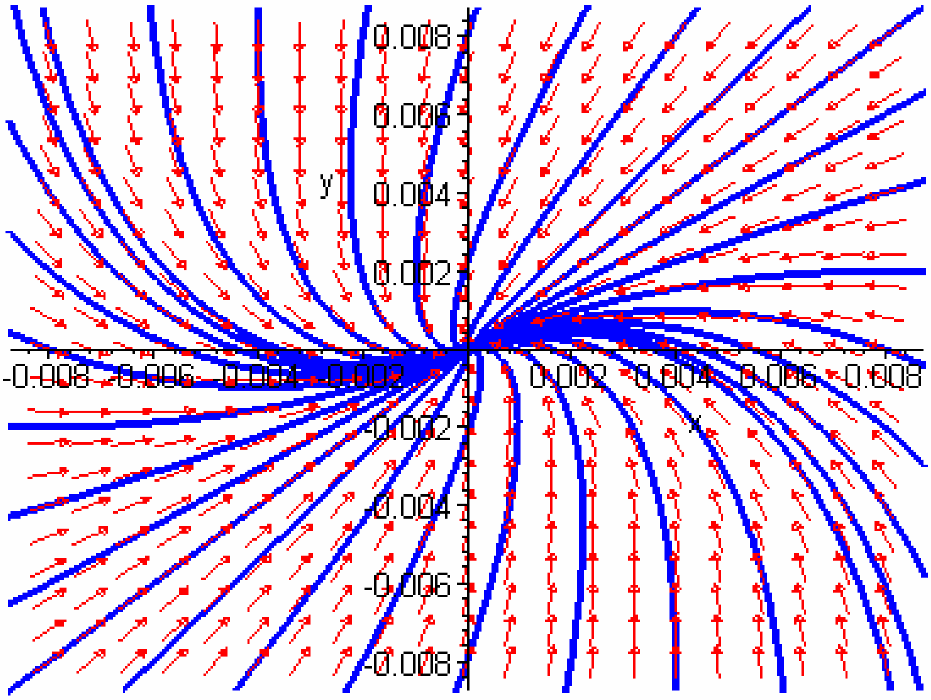
$$DE931 := \left[\frac{\partial}{\partial t} \mathrm{x}(t) = 2\sqrt{1+\mathrm{y}(t)} - 2\,\mathbf{e}^{(\mathrm{x}(t)+\mathrm{y}(t))}, \frac{\partial}{\partial t} \mathrm{y}(t) = \ln(1-4\,\mathrm{y}(t)) \right]$$



a=1

```
restart:with(DEtools):  a:=0.001:aa:=1:
DE931:=[diff(x(t),t)=sqrt(4+4*y(t))-2*exp(x(t)+y(t)),
diff(y(t),t)=sin(aa*x(t))+ln(1-4*y(t))];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=-8*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-6*a],
[x(0)=-8*a,y(0)=-4*a],[x(0)=-8*a,y(0)=-2*a],
[x(0)=-8*a,y(0)=0],[x(0)=-8*a,y(0)=2*a],
[x(0)=-8*a,y(0)=4*a],[x(0)=-8*a,y(0)=6*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=8*a,y(0)=-6*a],[x(0)=8*a,y(0)=-4*a],
[x(0)=8*a,y(0)=-2*a],[x(0)=8*a,y(0)=0],
[x(0)=8*a,y(0)=2*a],[x(0)=8*a,y(0)=4*a],
[x(0)=8*a,y(0)=6*a],[x(0)=8*a,y(0)=8*a],
[x(0)=-6*a,y(0)=-8*a],[x(0)=-4*a,y(0)=-8*a],
[x(0)=-2*a,y(0)=-8*a],[x(0)=0,y(0)=-8*a],
[x(0)=2*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=6*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-6*a,y(0)=8*a],[x(0)=-4*a,y(0)=8*a],
[x(0)=-2*a,y(0)=8*a],[x(0)=0,y(0)=8*a],
[x(0)=2*a,y(0)=8*a],[x(0)=4*a,y(0)=8*a],
[x(0)=6*a,y(0)=8*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```

$$DE931 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = 2\sqrt{1+\mathrm{y}(t)} - 2\,\mathrm{e}^{(\mathrm{x}(t)+\mathrm{y}(t))}, \frac{\partial}{\partial t}\mathrm{y}(t) = \sin(\mathrm{x}(t)) + \ln(1-4\,\mathrm{y}(t))\right]$$



a=10

```
 restart:with(DEtools):  a:=0.001:aa:=10:
DE931:=[diff(x(t),t)=sqrt(4+4*y(t))-2*exp(x(t)+y(t)),
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[x(0)=-8*a,y(0)=4*a],[x(0)=-8*a,y(0)=6*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=8*a,y(0)=-6*a],[x(0)=8*a,y(0)=-4*a],
[x(0)=8*a,y(0)=-2*a],[x(0)=8*a,y(0)=0],
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[x(0)=-2*a,y(0)=-8*a],[x(0)=0,y(0)=-8*a],
[x(0)=2*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=6*a,y(0)=-8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-6*a,y(0)=8*a],[x(0)=-4*a,y(0)=8*a],
[x(0)=-2*a,y(0)=8*a],[x(0)=0,y(0)=8*a],
[x(0)=2*a,y(0)=8*a],[x(0)=4*a,y(0)=8*a],
[x(0)=6*a,y(0)=8*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
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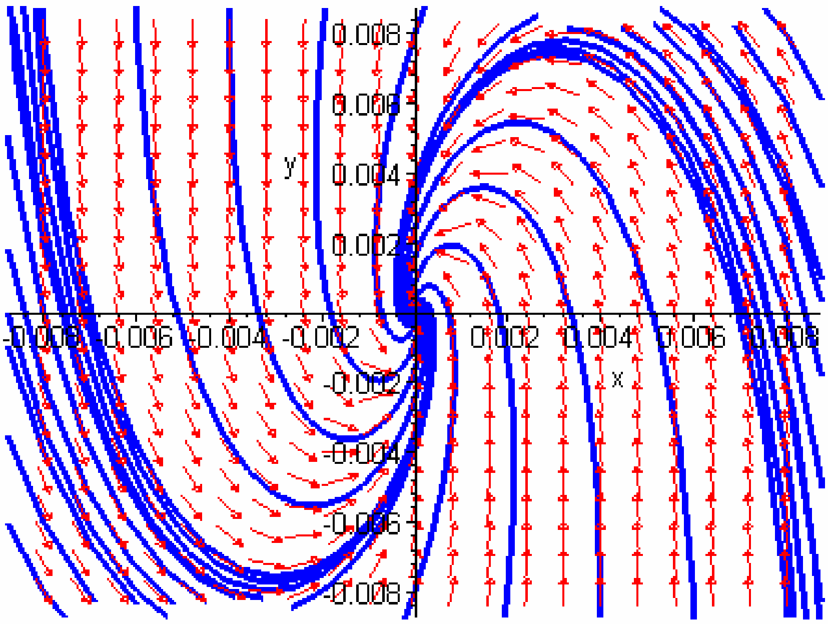
$$DE931 := \left[\frac{\partial}{\partial t} \mathrm{x}(t) = 2\sqrt{1+\mathrm{y}(t)} - 2\,\mathbf{e}^{(\mathrm{x}(t)+\mathrm{y}(t))}, \frac{\partial}{\partial t} \mathrm{y}(t) = \sin(10\,\mathrm{x}(t)) + \ln(1-4\,\mathrm{y}(t)) \right]$$



2.

1） $\lim\limits_{(x,y)\to(0,0)}\dfrac{y^2}{\sqrt{x^2+y^2}}=\lim\limits_{r\to 0}\dfrac{r^2\sin^2\theta}{r}=0$

奇点(0, 0)　(-1, 1)

$$\begin{cases}\dfrac{dx}{dt}=x\\ \dfrac{dy}{dt}=x+y\end{cases}\qquad \begin{vmatrix}1-\lambda & 0\\ 1 & 1-\lambda\end{vmatrix}=\lambda^2+2\lambda+1$$

$\Delta=0\quad p<0\quad q>0$ (0, 0)为不稳定的临界-退化结点

令 X=x+1， Y=y-1 代入则 $\begin{cases}\dfrac{dX}{dt}=X+2Y+Y^2\\ \dfrac{dY}{dt}=X+Y\end{cases}$ 可求得 p=1， q=-1， (-1, 1) 是鞍点

2). $\begin{cases}xy-1=0\\ x=y^3\end{cases}\qquad \begin{cases}x=1\\ y=1\end{cases}\qquad \begin{cases}x=-1\\ y=-1\end{cases}$

① 作平移变换，令 $X=x-1,\quad Y=y-1$，有

$$\begin{cases}\dfrac{dX}{dt}=1-(X+1)(Y+1)=1-X-Y-XY-1=-X-Y-XY\\ \dfrac{dY}{dt}=(X+1)-(Y+1)^3=X+1-Y^3-3y^2-3Y-1=X-Y^3-3Y^2-3Y\end{cases}$$

易证 $\lim\limits_{(x,y)\to(0,0)}\dfrac{XY}{\sqrt{X^2+Y^2}}=0\qquad \lim\limits_{(x,y)\to(0,0)}\dfrac{Y^3+3Y^2}{\sqrt{X^2+Y^2}}=0$

$$\therefore\ \begin{cases}\dfrac{dX}{dt}=-X-Y\\ \dfrac{dY}{dt}=X-3Y\end{cases}\qquad \begin{vmatrix}-1-\lambda & -1\\ 1 & -3-\lambda\end{vmatrix}=(\lambda+1)(\lambda+3)+1=\lambda^2+4\lambda+4$$

有(1, 1)处 $p>0\quad q>0\quad \Delta=0$ 稳定结点(退化-临界)

②在（-1，1）处，奇点是鞍点

3） $\begin{cases}x(1-x-y)=0\\ y(3-x-2y)=0\end{cases}\qquad \begin{cases}x=0\\ y=0\end{cases}\qquad \begin{cases}x=0\\ y=\dfrac{3}{2}\end{cases}\qquad \begin{cases}x=1\\ y=0\end{cases}\qquad \begin{cases}x=-1\\ y=2\end{cases}$

① （0，0）为奇点， 不稳定结点

② $(0,\dfrac{3}{2})$ 令 $Y=y-\dfrac{3}{2}$ 代入

$$\begin{cases}\dfrac{dX}{dt}=X-X^2-X(Y+\dfrac{3}{2})=\dfrac{-1}{2}X-X^2-XY\\ \dfrac{dY}{dt}=3(Y+\dfrac{3}{2})-X(Y+\dfrac{3}{2})-2(Y+\dfrac{3}{2})^2\\ \quad=3Y-\dfrac{3}{2}X+\dfrac{9}{2}-XY-2Y^2-\dfrac{9}{2}-6Y=-3Y-\dfrac{3}{2}X-XY-2Y^2\end{cases}$$

可判断 $(0,\dfrac{3}{2})$ 是稳定结点

③令 $X=x-1,\quad Y=y$

$$\begin{cases}\dfrac{dx}{dt}=(x+1)-(x+1)^2-(x+1)y=x+1-x^2-2x-1-xy-y\\ \quad=-x-y-x^2-xy\\ \dfrac{dy}{dt}=3y-(x+1)y-2y^2=3y-y-xy-2y^2=2y-xy-2y^2\end{cases}$$

可判断（1，0）是 鞍点

④令 $X=x+1,\quad Y=y-2$ 代入

$$\begin{cases}\dfrac{dX}{dt}=(X-1)-(X-1)^2-(X-1)(Y+2)\\ \quad=X-1-X^2+2X-1-XY2X+Y+2=+X+Y-XY-X^2\\ \dfrac{dY}{dt}=3(Y+2)-(X-1)(Y+2)-2(Y+2)^2\\ \quad=3Y+6-XY-2X+Y+2-2Y^2-8Y-8=-4Y-2X-XY-2Y^2\end{cases}$$

可判断（-1，2）是 鞍点

4）$\begin{cases}1-y=0\\ x^2-y^2=0\end{cases}$ $\quad\begin{cases}x=1\\ y=1\end{cases}$ $\quad\begin{cases}x=-1\\ y=1\end{cases}$

与前面方法类似，（1，1）稳定焦点；（-1，1） 鞍点

3．解：1）$\begin{cases}x(\gamma_1-\alpha_1x-\beta_1y)=0\\ y(\gamma_2-\alpha_2y-\beta_2x)=0\end{cases}$ $\quad\begin{cases}x=0\\ y=0\end{cases}$ $\quad\begin{cases}x=0\\ y=\dfrac{\gamma_2}{\alpha_2}\end{cases}$ $\quad\begin{cases}x=\dfrac{\gamma_1}{\alpha_1}\\ y=0\end{cases}$

正平衡点 $x^*=\dfrac{\alpha_2\gamma_1-\beta_1\gamma_2}{\alpha_1\alpha_2-\beta_1\beta_2}$ $\qquad y^*=\dfrac{\alpha_1\gamma_2-\beta_2\gamma_1}{\alpha_1\alpha_2-\beta_1\beta_2}$

2）代换验证即可（略）

3）令 $X=x-x^*\qquad Y=y-y^*$， 则代入即可知平衡点是稳定的结点

4．解：1）$\begin{cases}\dfrac{dx}{dt}=y\\ \dfrac{dy}{dt}=-C(x)y-g(x)\end{cases}$

2）$\begin{cases}y=0\\ -C(x)y-g(x)=0\end{cases}$　　　　$\begin{cases}x=0\\ y=0\end{cases}$

系统奇点为（0，0），令 $-C(x)y-g(x)=ax+by+\varphi(x,y)$，用泰勒公式

$-C(x)\in C^1(R),\quad g(x)\in C^2_{(R)}$

$-\left[C(0)+C^1(\xi)x\right]y-(g(0)+g'(0)x+\dfrac{g''(\theta)}{2!}x^2)=ax+by+\varphi(x,y)$

$\dfrac{dy}{dt}=-g^{'}(x)x-C(0)y+\varphi(x,y)$

易证　$\lim\limits_{(x,y)\to(0,0)}\varphi(x,y)=0$

$\therefore\ \begin{cases}\dfrac{dx}{dt}=y\\ \dfrac{dy}{dt}=-g'(0)x-C(0)y\end{cases}$

3）证明：$\begin{vmatrix}-\lambda & 1\\ -g'(0) & -C(0)-\lambda\end{vmatrix}=\lambda^2+C(0)\lambda+g'(0)$

$\Delta=C^2(0)-4g'(0)\qquad p=C(0),\quad q=g'(0)$

若 $C(0)>0,\ g'(0)>0$，即 $p>0,\quad q>0$，则奇点是渐近稳定点.

若 $C(0)<0$ 或 $g'(0)<0$，即 $p<0$ 或 $q<0$，奇点均为不稳定奇点.

5

证明:1）(充分性)

由于 $x(t)=\Phi(t)x(t_0)$

$\|\Phi(t)\|<K$，即　，$\forall\ \varepsilon>0$，$\exists\delta=\varepsilon/K>0,\quad \forall t\in[t_0,+\infty),\quad \|x(t_0)\|<\delta$,，

则有 $\|\Phi(t)x(t_0)\|<K\|x(t_0)\|<\varepsilon$，　所以零解稳定.

必要性：　设系统零解稳定，即 $\forall\varepsilon>0,\exists\delta>0,\quad \forall t\in[t_0,+\infty),\quad \|x(t_0)\|<\delta$，$\|\Phi(t)x(t_0)\|<\varepsilon$。若 $\|\Phi(t)\|<K$ 不成立，则对任意的自然数 m，必有 $t_m(t_m\to+\infty)$，使得

$\|\Phi(t_m)\| > m$，这与$\|\Phi(t_m)x(t_0)\| < \varepsilon$ 矛盾。

2）系统零解渐近稳定等价于零解稳定，且

$$\lim_{t\to+\infty}\|\Phi(t)x(t_0)\| = 0 \Leftrightarrow \lim_{t\to+\infty}\|\Phi(t)\| = 0$$

6

解：1）令 $y' = y_2 \quad y'' = y_3 \quad y_1 = y$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -b & -a \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

特征方程$\begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -b & -a-\lambda \end{pmatrix} = -\lambda^2(a+\lambda) - 2 - b\lambda = -(\lambda^3 + a\lambda^2 + b\lambda + 2)$

$$\Delta_3 = \begin{pmatrix} a & 1 & 0 \\ 2 & b & a \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = a \qquad \Delta_2 = ab - 2 \qquad \Delta_3 = 2ab - 4 = 2(ab-2)$$

若 $a > 0, ab - 2 > 0$ 零解稳定，若 $a < 0$或$ab - 2 < 0$ 零解不稳定.

2）令 $y_1 = y \quad y_2 = y' = y_1' \quad y_3 = y'' = y_2' \quad y_4 = y''' = y_3'$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & 0 & -3 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

特征方程是：$\lambda^4 + 2\lambda^3 + 3\lambda^2 + a$

$$\Delta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & a & 0 & 3 \\ 0 & 0 & 0 & a \end{pmatrix}$$

$$\Delta_1 = 2 > 0 \qquad \Delta_2 = 6 > 0 \qquad \Delta_3 = -4a$$

$$\Delta_4 = a\begin{vmatrix} 2 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & a & 0 \end{vmatrix} = a(-4a) = -4a^2 \le 0$$

$\therefore$ 零解不稳定

3）原方程组在（0，0）点处的线性近似方程组的系数矩阵为

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$ 特征方程是 $1 + 3\lambda - \lambda^3 = 0$

$F(0) = 0$ 且 $\lim\limits_{\rho \to 0} \dfrac{|F(\rho)|}{\rho} = 0$

$$\Delta = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\Delta_1 = 0 \qquad \Delta_2 = 1 \qquad \Delta_3 = -1$

$\therefore$ 零解不稳定

4）$F(0) = 0$ 且 $\lim\limits_{\rho \to 0} \dfrac{|F(\rho)\|}{\rho} = 0$

$$|A - \lambda| = \begin{pmatrix} -\lambda & -2 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = \lambda^2(1-\lambda) + 2(1-\lambda) = (\lambda^2 + 2)(1-\lambda) = 0$$

$\lambda = 1 \qquad \lambda = \pm\sqrt{2}\, i$

非线性系统显然是不稳定的.