答案 5

1.

1) $5x-3x^2=0$ $x=\frac{5}{3}$ 是稳定的， x=0 不稳定.

2) $x=0,$ $x=\frac{1}{2},$ $x=2$

$x=0, x=2$ 为稳定解，$x=\frac{1}{2}$ 为不稳定解

3) $x=0$ 不稳定， $x=k$ 稳定，

4) $x=0,$ $x=3$ $x=0$ 解稳定，$x=3$ 解不稳定.

2.

1）a<0 稳定,a>0 不稳定

2）不稳定

3）稳定

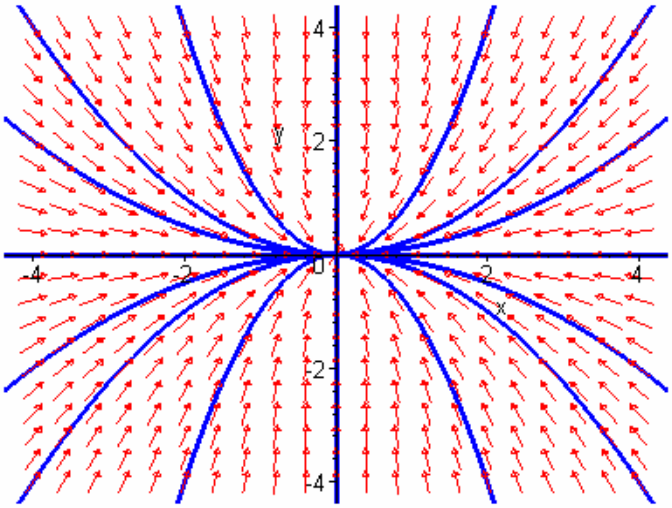
4）$\lambda>0$，不稳定；$\lambda<0$，稳定

3.

1）渐近稳定， $y=cx^2$

```
> restart:with(DEtools):  a:=1/2:
DE931:=[diff(x(t),t)=-x(t),
diff(y(t),t)=-2*y(t)];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=0,y(0)=8*a],[x(0)=0,y(0)=-8*a],
[x(0)=8*a,y(0)=0],[x(0)=-8*a,y(0)=0],
[x(0)=8*a,y(0)=8*a],[x(0)=-8*a,y(0)=-8*a],
[x(0)=-8*a,y(0)=8*a],[x(0)=8*a,y(0)=-8*a],
[x(0)=-4*a,y(0)=8*a],[x(0)=4*a,y(0)=8*a],
[x(0)=-4*a,y(0)=-8*a],[x(0)=4*a,y(0)=-8*a],
[x(0)=-8*a,y(0)=4*a],[x(0)=8*a,y(0)=4*a],
[x(0)=-8*a,y(0)=-4*a],[x(0)=8*a,y(0)=-4*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```

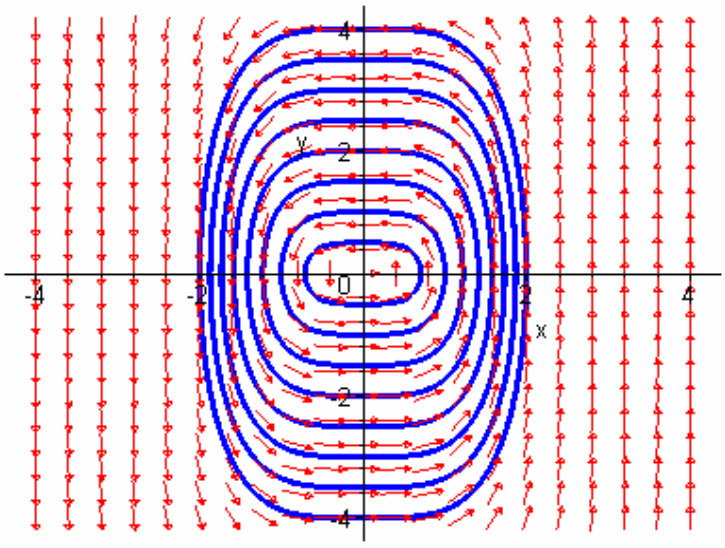
$$DE931 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = -\mathrm{x}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = -2\,\mathrm{y}(t)\right]$$



2）稳定， $y^2 = c - x^4$

```
> restart:with(DEtools):  a:=1/2:
DE931:=[diff(x(t),t)=-y(t),
diff(y(t),t)=2*x(t)^3];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=0,y(0)=8*a],[x(0)=0,y(0)=7*a],
[x(0)=0,y(0)=6*a],[x(0)=0,y(0)=5*a],
[x(0)=0,y(0)=4*a],[x(0)=0,y(0)=3*a],
[x(0)=0,y(0)=2*a],[x(0)=0,y(0)=1*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```

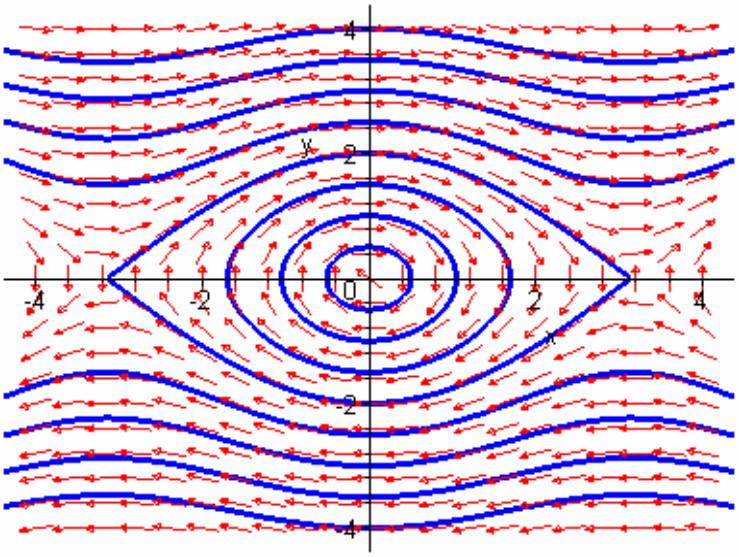
$$DE931 := \left[\frac{\partial}{\partial t} \mathrm{x}(t) = -\mathrm{y}(t), \frac{\partial}{\partial t} \mathrm{y}(t) = 2\,\mathrm{x}(t)^3 \right]$$



3）稳定，$y^2 = c + \frac{1}{2}\cos x$

```
 restart:with(DEtools):  a:=1/2:
DE931:=[diff(x(t),t)=y(t),
diff(y(t),t)=-sin(x(t))];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=0,y(0)=8*a],[x(0)=0,y(0)=7*a],
[x(0)=0,y(0)=6*a],[x(0)=0,y(0)=5*a],
[x(0)=0,y(0)=-8*a],[x(0)=0,y(0)=-7*a],
[x(0)=0,y(0)=-6*a],[x(0)=0,y(0)=-5*a],
[x(0)=0,y(0)=4*a],[x(0)=0,y(0)=3*a],[x(0)=0,y(0)=-4*a],
[x(0)=0,y(0)=6*a],[x(0)=0,y(0)=5*a],
[x(0)=0,y(0)=2*a],[x(0)=0,y(0)=1*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```

$$DE931 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = -\sin(\mathrm{x}(t))\right]$$



4）不稳定，$1 + y^2 = ce^{x^4/2}$

```
 > restart:with(DEtools):  a:=1/5:
DE931:=[diff(x(t),t)=y(t),
diff(y(t),t)=x(t)^3*(1+y(t)^2)];
DEplot(DE931,[x(t),y(t)], t=-10..10,
[[x(0)=0,y(0)=8*a],[x(0)=0,y(0)=7*a],
[x(0)=0,y(0)=6*a],[x(0)=0,y(0)=5*a],
[x(0)=0,y(0)=-8*a],[x(0)=0,y(0)=-7*a],
[x(0)=0,y(0)=-6*a],[x(0)=0,y(0)=-5*a],
[x(0)=0,y(0)=4*a],[x(0)=0,y(0)=3*a],
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[x(0)=0,y(0)=-4*a],[x(0)=0,y(0)=-2*a],
[x(0)=0,y(0)=-2*a],[x(0)=0,y(0)=-1*a],
[x(0)=0,y(0)=-0.4*a],[x(0)=0,y(0)=0.5*a],
[x(0)=0,y(0)=6*a],[x(0)=0,y(0)=5*a],
[x(0)=0,y(0)=2*a],[x(0)=0,y(0)=1*a]],
x=-8*a..8*a,y=-8*a..8*a, stepsize=0.05,
dirgrid=[21,21], color=red,linecolor=blue,
arrows=SLIM);
```
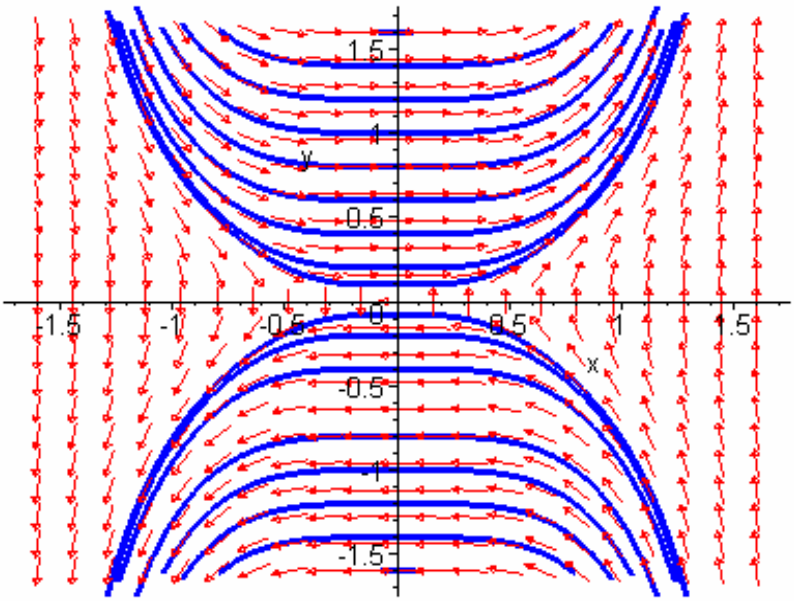

$$DE931 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = \mathrm{x}(t)^3\,(1+\mathrm{y}(t)^2)\right]$$



4.

1） $$\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{r^2(\cos^2\theta-\sin^2\theta)}{r} = 0$$

$$\lim_{(x,y)\to(0,0)}\frac{x^2 y}{\sqrt{x^2+y^2}} = \lim_{r\to 0}\frac{r^2\cos^2\theta r.\sin^2\theta}{r} = 0$$

∴ 方程组

$$\begin{cases}\dfrac{dx}{dt} = x-y\\[2ex] \dfrac{dy}{dt} = x+y\end{cases} \qquad \begin{vmatrix}1-\lambda & -1\\ 1 & 1-\lambda\end{vmatrix} = \lambda^2+1-2\lambda+1 = \lambda^2-2\lambda+2$$

$p<0,\quad q>0,\quad \Delta<0$ 不稳定焦点

2） $$\lim_{(x,y)\to(0,0)}\frac{xy}{\sqrt{x^2+y^2}} = 0, \qquad \lim_{(x,y)\to(0,0)}\frac{x^4+y^3}{\sqrt{x^2+y^2}} = 0$$

$$\begin{cases}\dfrac{dx}{dt}=-x+y\\ \dfrac{dy}{dt}=2x-3y\end{cases}\qquad p>0,\quad q>0,\quad \Delta=12>0\quad \text{稳定结点}$$

3） $e^{y}-1\sim y,\qquad \cos y-1\sim -\dfrac{y^{2}}{2},\qquad \sin y\sim y$

$$\therefore\quad \begin{cases}\dot{x}=x+(y+1)-(1-\dfrac{y^{2}}{2})=x+y+\dfrac{y^{2}}{2}\\ \dot{y}=3x-2y\end{cases}\qquad \lim_{(x,y)\to(0,0)}\frac{\frac{y^{2}}{2}}{\sqrt{x^{2}+y^{2}}}=0$$

$$\begin{cases}\dot{x}=x+y\\ \dot{y}=3x-2y\end{cases}\qquad p>0,\quad q<0,\quad \Delta>0\quad \text{鞍点}$$

4) $\lim\limits_{(x,y)\to(0,0)}\dfrac{\sin x-x}{\sqrt{x^{2}+y^{2}}}=0$ $\qquad\begin{cases}\dot{x}=y+x\\ \dot{y}=ax+by\end{cases}$

$p=-(b+1),\quad q=(b-a),\quad \Delta=(b+1)^{2}-4(b-a)=(b-1)^{2}+4a$

$b<a$ 鞍点

$b>a,\quad b<-1,$ △>0，稳定结点；△<0，稳定焦点；

$b>a,\quad b>-1,$ △>0，不稳定结点；△<0，不稳定焦点

5) $\lim\limits_{(x,y)\to(0,0)}\dfrac{y^{2}}{\sqrt{x^{2}+y^{2}}}=0$ $\qquad\lim\limits_{(x,y)\to(0,0)}\dfrac{x^{2}}{\sqrt{x^{2}+y^{2}}}=0$

$$\begin{cases}\dot{x}=x+ay\\ \dot{y}=bx-3y\end{cases}\qquad p>0,\quad q=-(3+ab),\quad \Delta=4+4(ab+3)=4(ab+4)$$

$ab<-4$ 稳定结点

$-4<ab<-3$ 稳定焦点

$ab>-3$ 鞍点

5.

1) $V(x,y)=\dfrac{1}{2}(x^{2}+y^{2})$ 渐近稳定

2）$V(x,y)=\dfrac{1}{2}(x^{2}+y^{2})$ 渐近稳定

3）$V(x,y)=\dfrac{1}{2}(x^{2}+y^{2})$ 不稳定

4）$V(x,y)=\dfrac{1}{2}(x^{2}+y^{2})$

$\alpha<0$ 渐近稳定

$\alpha = 0$　　稳定

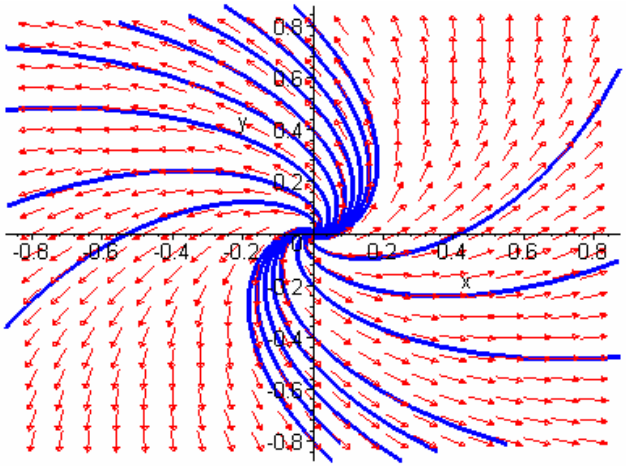
$\alpha > 0$　　不稳定

6.

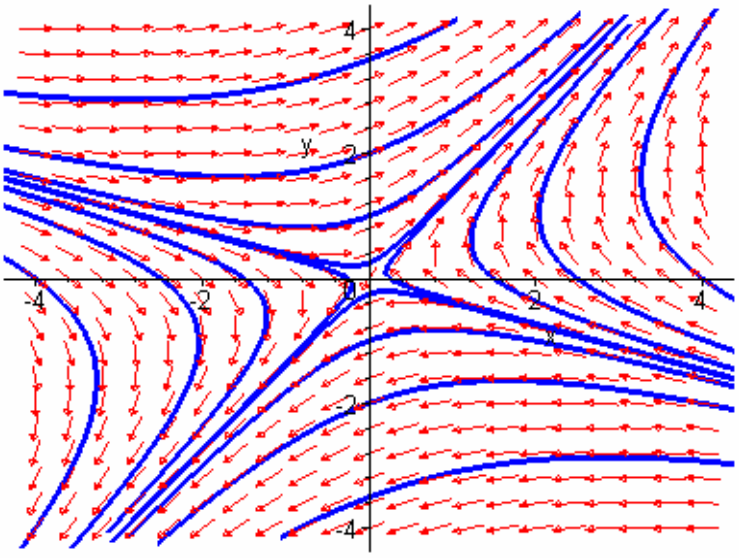
$\frac{dx}{dt} = f(x) = f(0) + f'(0)x + o(x) = f'(0)x + o(x)$，　零点是稳定的

7.

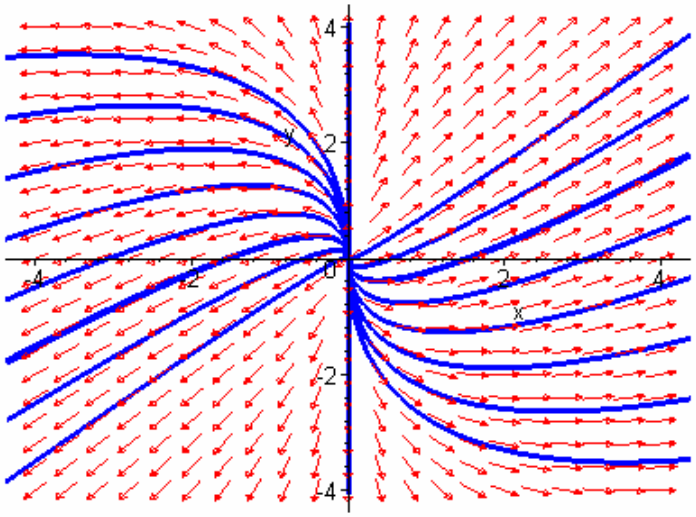
1）不稳定焦点



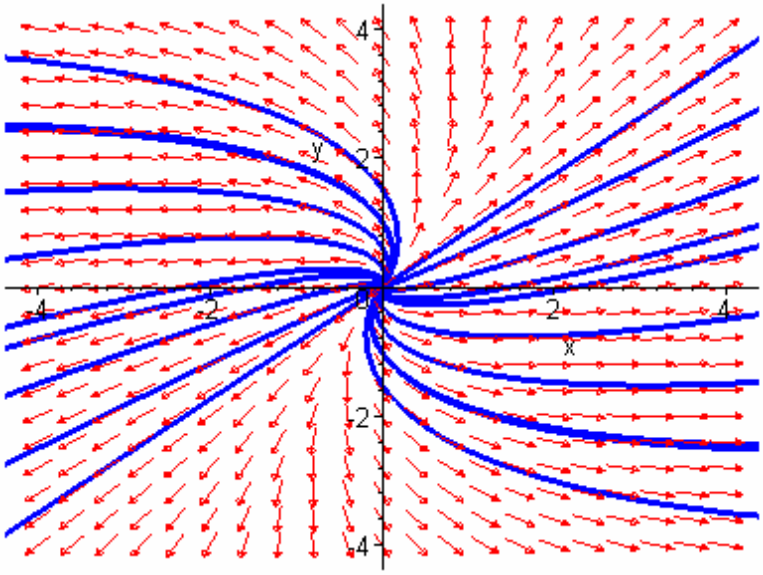
2）鞍点



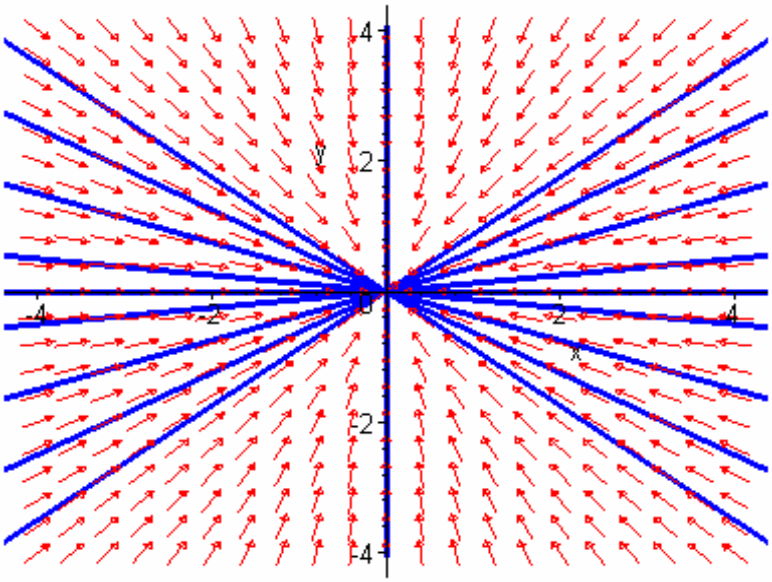
3）不稳定结点



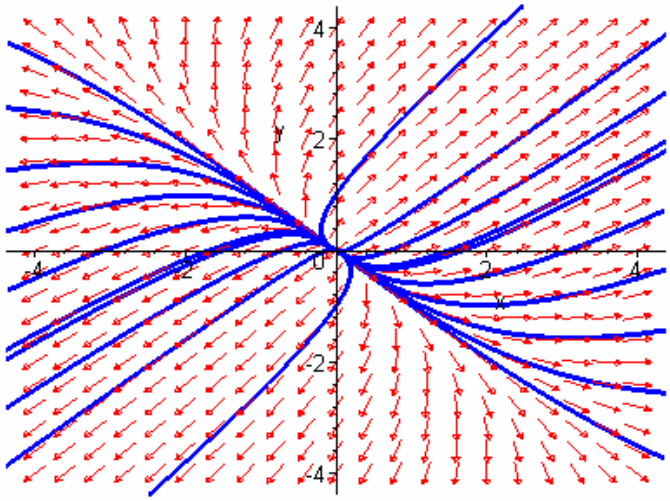
4）不稳定的退化结点



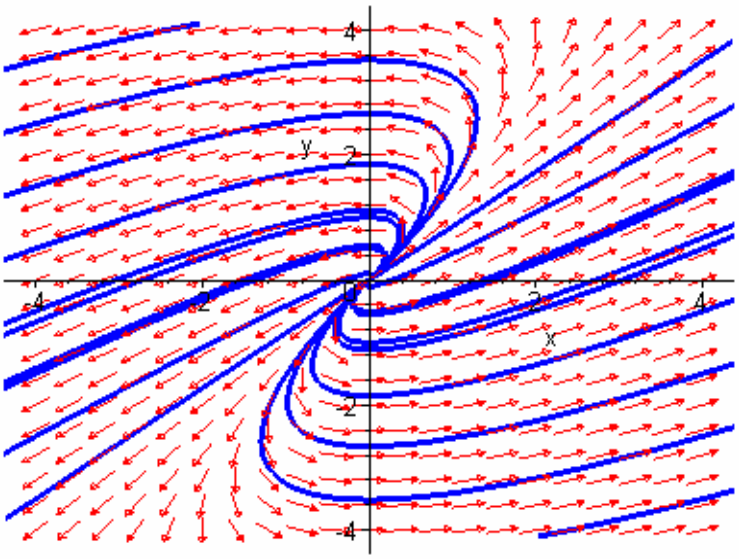
5）稳定临界结点



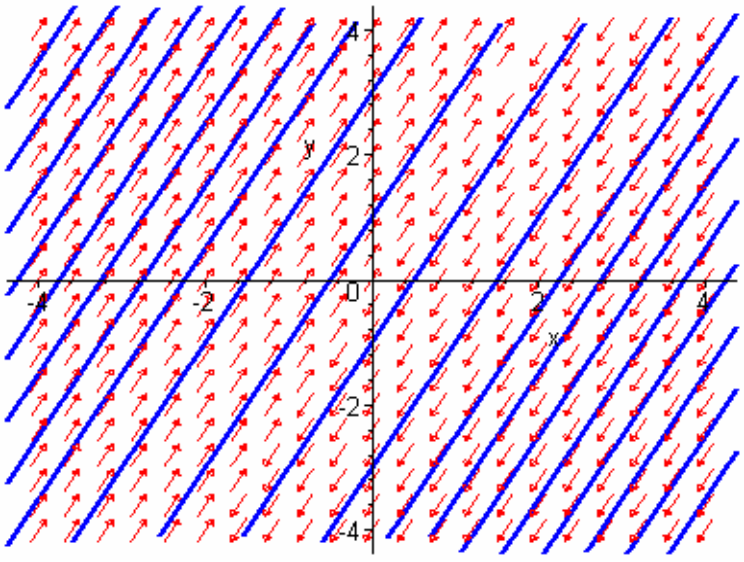
6) 不稳定结点



7）不稳定的退化结点



8) 奇线



8.解： $\begin{vmatrix} a-\lambda & b \\ 0 & c-\lambda \end{vmatrix} = (\lambda - a)(\lambda - c) = \lambda^2 - (a+c)\lambda + ac$

$p = -(a+c) \quad q = ac \quad \Delta = (a+c)^2 - 4ac = (a-c)^2$

① $q < 0$ 鞍点

② $p > 0 \quad q > 0 \quad \begin{cases} \Delta > 0 \\ \Delta = 0 \end{cases}$ 稳定结点/ 稳定临界-退化结点

$p < 0 \quad q > 0 \quad \begin{cases} \Delta > 0 \\ \Delta = 0 \end{cases}$ 不稳定结点/ 不稳定临界-退化结点

③ $p = 0 \quad q > 0$ 中心

9.证明： $\begin{vmatrix} m-\lambda & n \\ -a & -b-\lambda \end{vmatrix} = (\lambda + b)(\lambda - m) + an = \lambda^2 - m\lambda + b\lambda + an - bm$

$p = (b - m) = 0 \qquad q = an - bm$

$q > 0, \quad p = 0, \quad m = b$ 时 $\dfrac{\partial(ax+by)}{\partial y} = b = m = \dfrac{\partial(mx+ny)}{\partial x}$

$\therefore \quad (ax+by)dx + (mx+ny)dy = 0$

为全微分方程，反之，若 $b = m$，但 $an - bm < 0$ 时，（0，0）为鞍点，不是中心.

10.解： $\dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = -kx + my - y^3$ 仅有奇点(0,0)，

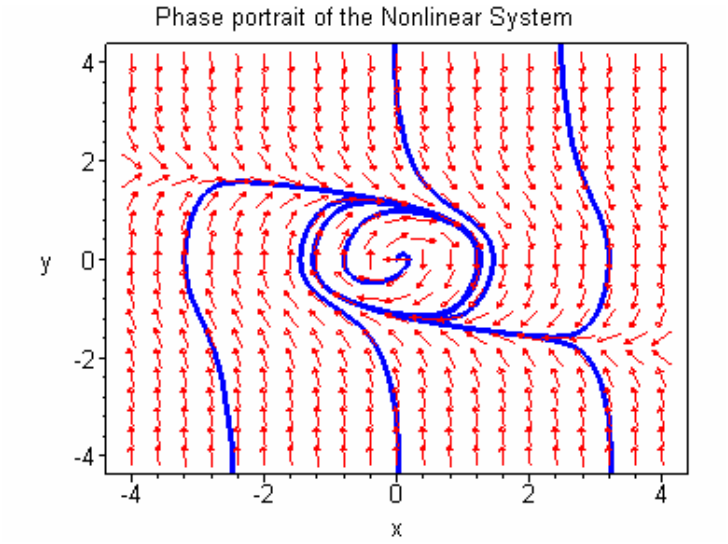
p=-m, q=k>0,

$p = m, \quad q = k > 0, \quad \Delta = m^2 - 4k$， (0,0)是稳定的结点或稳定焦点

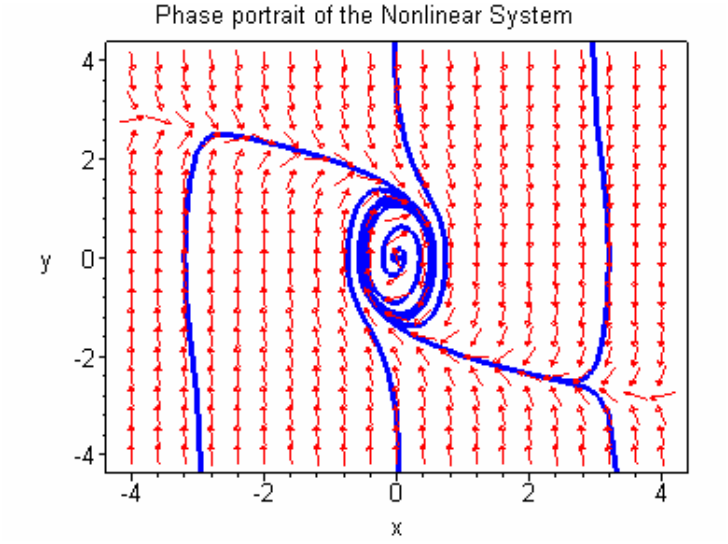
```
restart:with(DEtools): a:=4:k:=1: m:=1:
 ODE1 :=[diff(x(t),t)=y(t),
diff(y(t),t)=-k*x(t)+m*y(t)-y(t)^3];
DEplot( ODE1, [x(t),y(t)],t=-100..100,
[[x(0)=0.1,y(0)=0.1],[x(0)=0,y(0)=0.8*a],
[x(0)=0,y(0)=-0.8*a],[x(0)=0.8*a,y(0)=-0.8*a],
[x(0)=-0.8*a,y(0)=0],[x(0)=0.8*a,y(0)=0]],
 x=-a..a,y=-a..a,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Phase portrait of the Nonlinear System",
arrows=SLIM);
```

$$ODE1 := \left[\frac{\partial}{\partial t} x(t) = y(t), \frac{\partial}{\partial t} y(t) = -x(t) + y(t) - y(t)^3 \right]$$

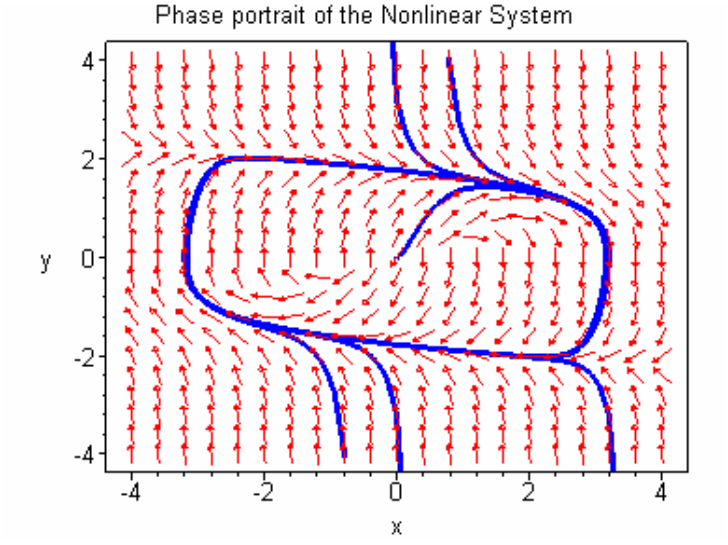
K=1,m=1



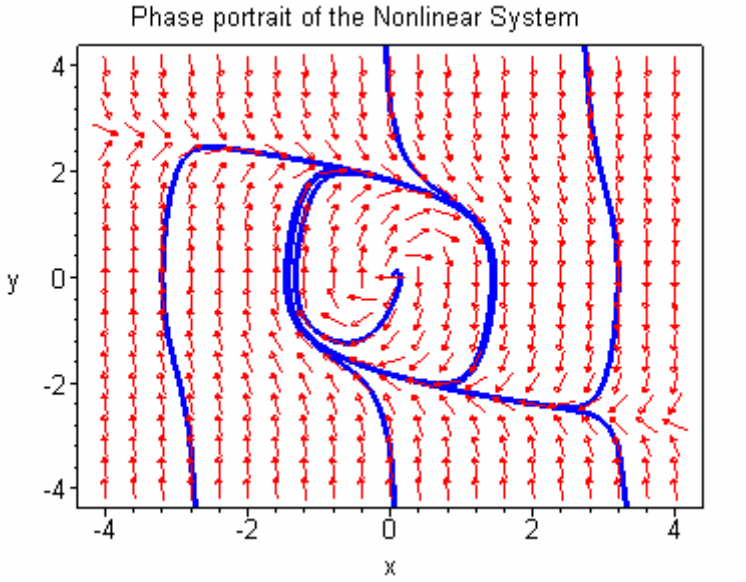
k=5,m=1



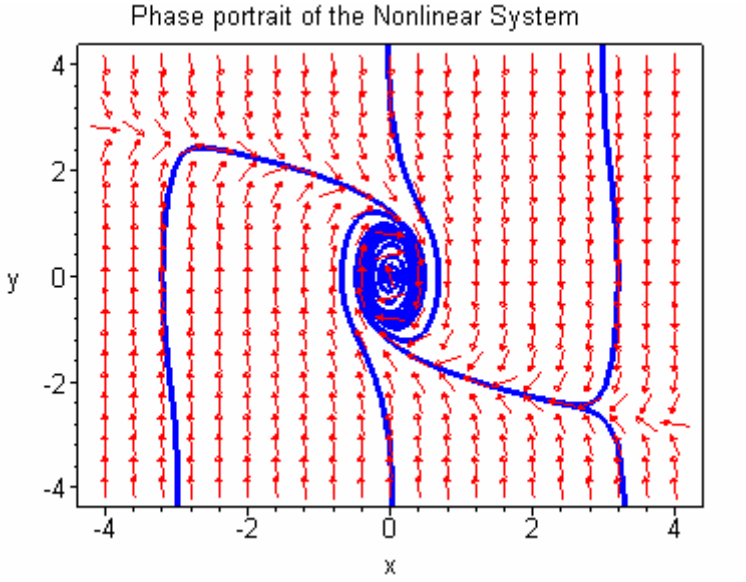
k=1,m=3



k=3,m=3



k=5,m=1/2



11.

1) 令 $x = r\cos\theta,\quad y = r\sin\theta$

$$\begin{cases} \dfrac{dr}{dt} = r(r-1) \\ \dfrac{d\theta}{dt} = r \end{cases}$$

$r = 1$ 是不稳定的极限环

2）$r \to 1$ 时，$r(1-r)\sin\dfrac{1}{1-r} \to 0,\quad \dfrac{dr}{dt} = r(1-r)\sin\dfrac{1}{1-r}$ 在 r=1 的邻域有无限多个极限环，相邻的两个有不同的稳定性。

12.

证明： 令 $x = r\cos\theta,\quad y = r\sin\theta$

$$\begin{cases} \dfrac{dr}{dt} = r\sin r^2 \\ \dfrac{d\theta}{dt} = 1 \end{cases}$$

$r = k\pi \qquad \theta = \theta_0 + (t - t_0)$)

是系统的孤立的极限环，k 为偶数时不稳定， k 为奇数时稳定。