答案5.1

1.

(1) 解, $\frac{dx}{dt}=x(r-sx)$

① 当 $x(0)=0$, 则 $x(t,0,0)=0$, $t\in(-\infty,\infty)$

② 当 $x(0)=\frac{r}{s}$, 则 $x(t,0,0)=\frac{r}{s}$, $t\in(-\infty,\infty)$

③ 当 $x(0)\neq 0,\frac{r}{s}$, 则解此方程得： $x=\dfrac{r}{s(1+\frac{r-sx(0)}{sx(0)}e^{-rt})}$,

a:

$x(0)\in(-\infty,0),\quad t\in(-\infty,\frac{1}{r}\ln(1-\frac{r}{sx(0)})),\quad \lim\limits_{t\to-\infty}x(t)=0,\quad \lim\limits_{t\to\frac{1}{r}\ln(1-\frac{r}{sx(0)})}x(t)=-\infty$

b: $x(0)\in(0,r/s),\quad t\in(-\infty,+\infty),\quad \lim\limits_{t\to-\infty}x(t)=0,\quad \lim\limits_{t\to+\infty}x(t)=r/s$

c:

$x(0)\in(r/s,+\infty),\quad t\in(\frac{1}{r}\ln(1-\frac{r}{sx(0)}),+\infty),\quad \lim\limits_{t\to\frac{1}{r}\ln(1-\frac{r}{sx(0)}),}x(t)=+\infty,\quad \lim\limits_{t\to+\infty}x(t)=r/s$

当 $r>0,\ s>0,\quad x(0)>0$ 时，从解的形式看出 $\lim\limits_{t\to+\infty}x(t,0,x_0)=\frac{r}{s}$

（2）解：方程奇点为 $x(t)=0,x(t)=1,x(t)=2$

① 若 $x(0)=0$, 则 $x(t,0,0)=0$

② 若 $x(0)=1$, 则 $x(t,0,1)=1$

③ 若 $x(0)=2$, 则 $x(t,0,2)=2$

④ 若 $x(0)\neq 0,1,2$ 时，

a) 当 $0<x(0)<1,\quad t\in(-\infty,+\infty),\quad \lim\limits_{t\to-\infty}x(t)=0,\quad \lim\limits_{t\to+\infty}x(t)=1,$

b) 当 $1<x(0)<2,\quad t\in(-\infty,+\infty),\quad \lim\limits_{t\to-\infty}x(t)=1,\quad \lim\limits_{t\to+\infty}x(t)=2,$

c) 直接解方程知

$$x(t)=1+\sqrt{1+\frac{e^{2t}}{\frac{(x_0-1)^2}{x_0(x_0-2)}-e^{2t}}},\quad x(0)=x_0>2,\quad T=\frac{1}{2}\ln\left(\frac{(x_0-1)^2}{x_0(x_0-2)}\right),$$
$$t\in(-\infty,T),\quad \lim_{t\to-\infty}x(t)=2,\quad \lim_{t\to T}x(t)=+\infty.$$

（3）方程的奇点为0，+1，-1。

① 若 $x(0)=-1,\ x(t,0,-1)=-1$

② 若 $x(0)=0,\ x(t,0,0)=0$

③ 若 $x(0)=1,\ x(t,0,1)=1$

④ 若 $x(0)\neq 0,\pm 1$ 时，

a: 当 $0<x(0)<1,\quad t\in(-\infty,+\infty),\quad \lim_{t\to-\infty}x(t)=1,\quad \lim_{t\to+\infty}x(t)=0,$

b: 当 $-1<x(0)<0,\quad t\in(-\infty,+\infty),\quad \lim_{t\to-\infty}x(t)=0,\quad \lim_{t\to+\infty}x(t)=-1,$

c: 解方程知，当

$x(0)=0>1,\quad x(t)$单调增， 方程的隐式解为

$$t+\frac{1}{x_0}-\frac{1}{x(t)}+\frac{1}{2}\ln\frac{(x(t)+1)(x_0-1)}{(x(t)-1)(x_0+1)}=0,$$

$$\lim_{x\to+\infty}\left(t+\frac{1}{x_0}-\frac{1}{x(t)}+\frac{1}{2}\ln\frac{(x(t)+1)(x_0-1)}{(x(t)-1)(x_0+1)}\right)=0,$$

$$\lim_{x\to+\infty}t=-\frac{1}{x_0}-\frac{1}{2}\ln\frac{(x_0-1)}{(x_0+1)}=\frac{1}{2}\ln\frac{(x_0+1)}{(x_0-1)}-\frac{1}{x_0},$$

由于$\frac{1}{2}\ln\frac{(x_0+1)}{(x_0-1)}-\frac{1}{x_0}$关于$x_0$单调减，$\frac{1}{2}\ln\frac{(x_0+1)}{(x_0-1)}-\frac{1}{x_0}\to 0,$

所以$T=\frac{1}{2}\ln\frac{(x_0+1)}{(x_0-1)}-\frac{1}{x_0}>0,\quad \lim_{t\to T}x(t)=+\infty,$

$t\in(-\infty,T),\quad \lim_{t\to-\infty}x(t)=1.$

d: 同理得，当 $x(0)<-1,\quad t\in(T,+\infty),\quad \lim_{t\to T}x(t)=-\infty,\quad \lim_{t\to+\infty}x(t)=-1,$

2．解: $\dfrac{dx}{dt}=a(t)x \qquad x(t)=x(0)e^{\int_0^t a(s)ds}$

零解稳定的充分必要条件为 $\int_0^t a(s)ds < M$

零解渐近稳定的充分必要条件为

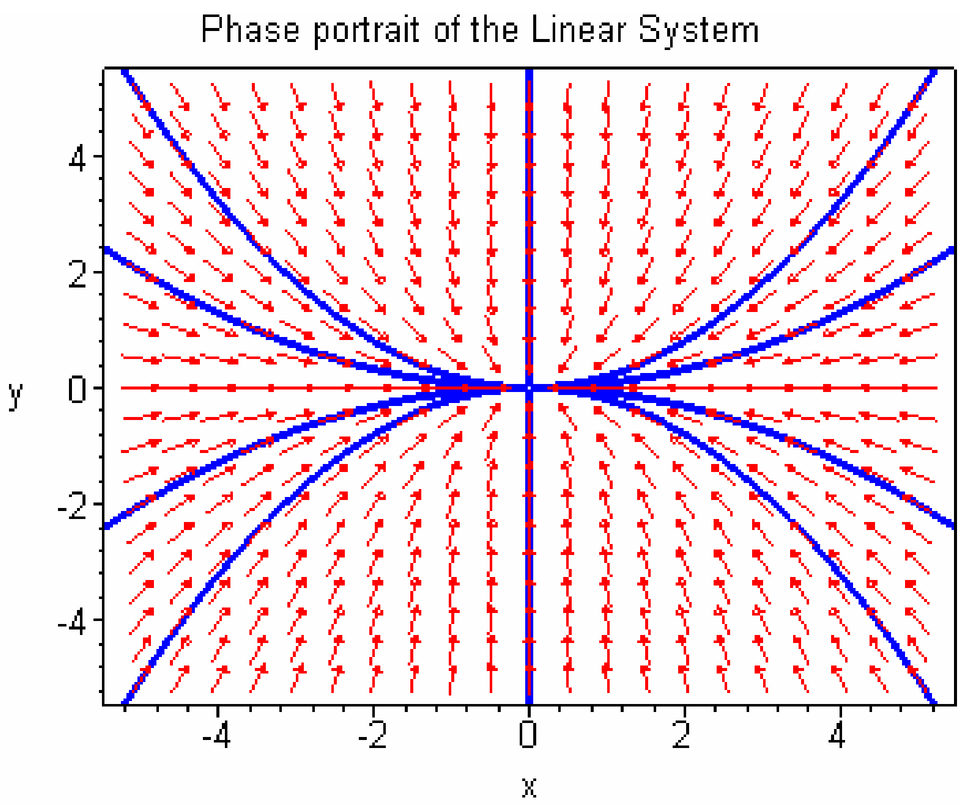
$$\int_0^t a(s)ds < M, \quad \int_0^t a(s)ds \xrightarrow[t\to+\infty]{} -\infty$$

3．图形可以参见书中**例5.1.5**的maple程序

（1） $y = cx^2$，零解渐近稳定

```
> restart:with(DEtools):
ODE1 :=[diff(x(t),t)=-x(t), diff(y(t),t)=-2*y(t)];
DEplot( ODE1, [x(t),y(t)],t=-10..10,
[[x(0)=0,y(0)=5],[x(0)=0,y(0)=-5],
[x(0)=5,y(0)=2],[x(0)=5,y(0)=5],[x(0)=5,y(0)=-2],[x(0)=5,
y(0)=-5],
[x(0)=-5,y(0)=2],[x(0)=-5,y(0)=5],[x(0)=-5,y(0)=-2],[x(0)
=-5,y(0)=-5]],
 x=-5..5,y=-5..5,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Phase portrait of the Linear System",
arrows=SLIM);
```

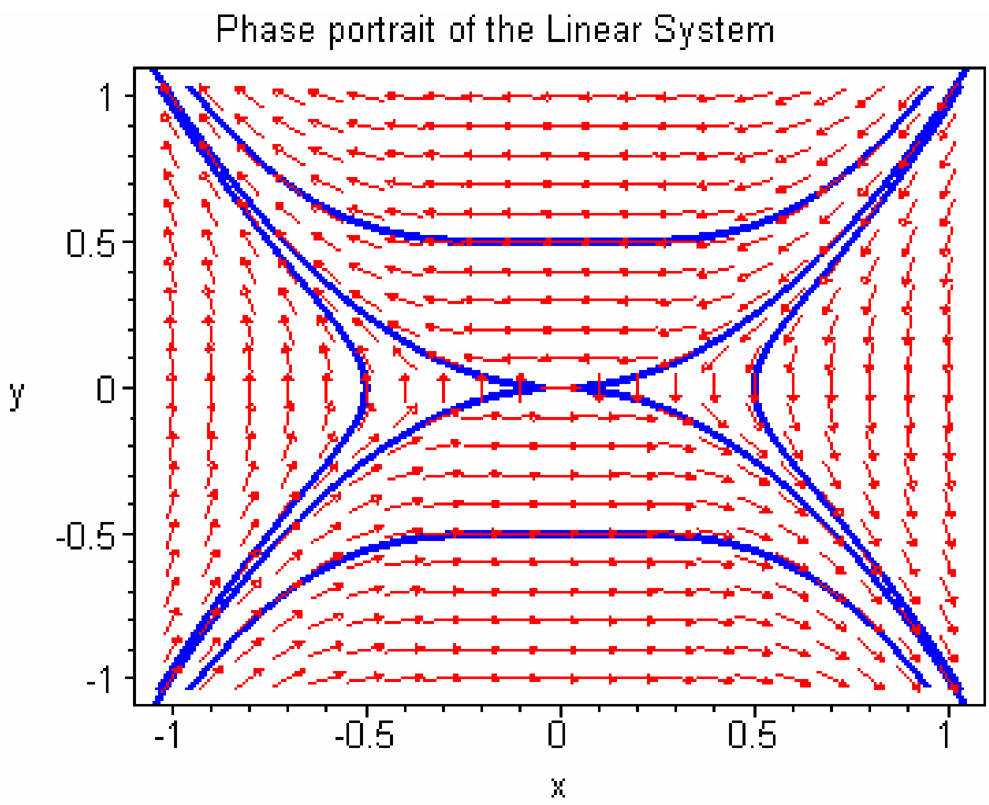
$$ODE1 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = -\mathrm{x}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = -2\,\mathrm{y}(t)\right]$$



(2) $y^2 = c + x^4$，零解不稳定

```
restart:with(DEtools): a:=1:
ODE1 :=[diff(x(t),t)=-y(t), diff(y(t),t)=-2*x(t)^3];
DEplot( ODE1, [x(t),y(t)],t=-100..100,
[[x(0)=0,y(0)=a/2],[x(0)=0,y(0)=-a/2],
[x(0)=a/2,y(0)=0],[x(0)=-a/2,y(0)=0],
[x(0)=a,y(0)=a],[x(0)=a,y(0)=-a],
[x(0)=-a,y(0)=a],[x(0)=-a,y(0)=-a]],
 x=-a..a,y=-a..a,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Phase portrait of the Linear System",
arrows=SLIM);
```

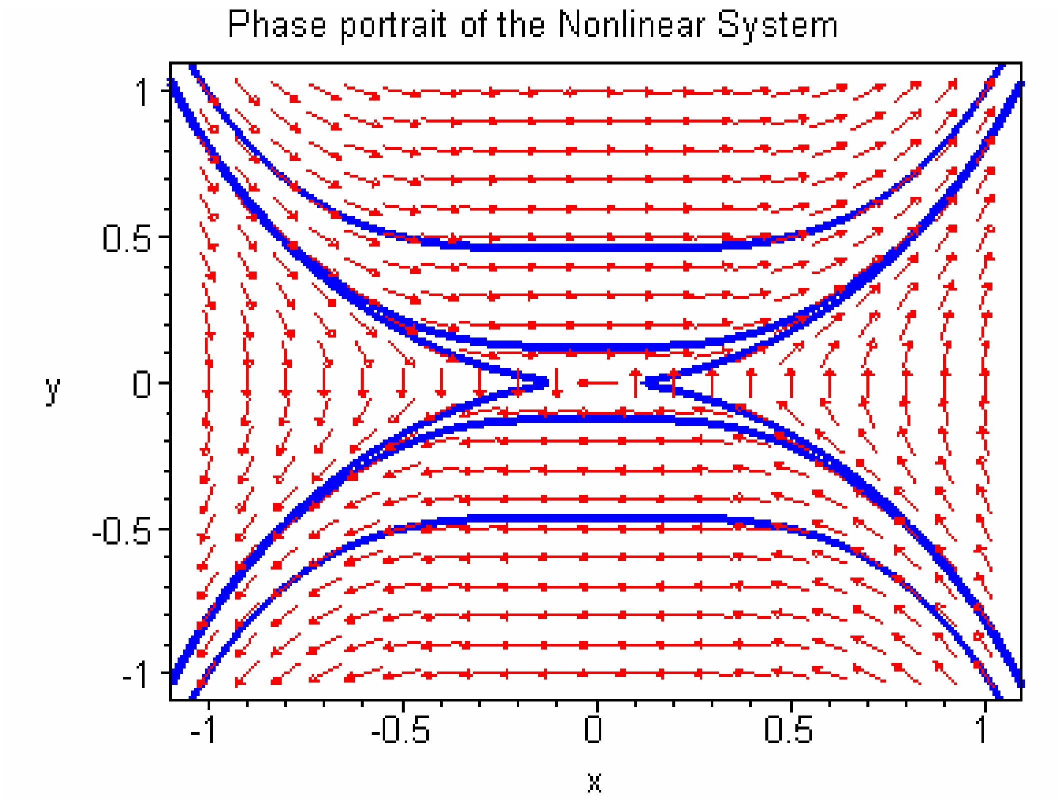
$$ODE1 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = -\mathrm{y}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = -2\,\mathrm{x}(t)^3\right]$$



（3） $y^2 = 1 - ce^{x^4/2}$，零解不稳定

```
restart:with(DEtools): a:=1:
ODE1 :=[diff(x(t),t)=y(t), diff(y(t),t)=x(t)^3*(1+y(t)^2)];
DEplot( ODE1, [x(t),y(t)],t=-100..100,
[[x(0)=0,y(0)=a/8],[x(0)=0,y(0)=-a/8],
[x(0)=a/8,y(0)=0],[x(0)=-a/8,y(0)=0],
[x(0)=a,y(0)=a],[x(0)=a,y(0)=-a],
[x(0)=-a,y(0)=a],[x(0)=-a,y(0)=-a]],
 x=-a..a,y=-a..a,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Phase portrait of the Nonlinear System",
arrows=SLIM);
```

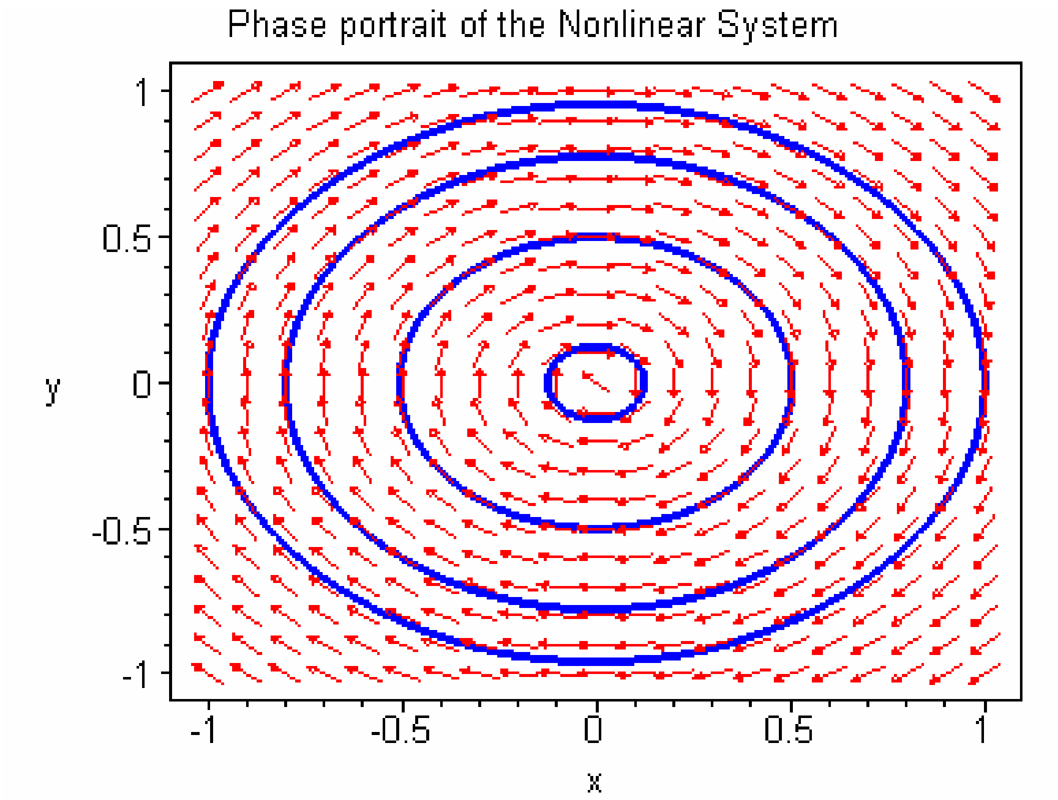
$$ODE1 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = \mathrm{x}(t)^3\,(1 + \mathrm{y}(t)^2)\right]$$



（4） $y^2 = c + \cos x$，零解稳定

```
 restart:with(DEtools): a:=1:
ODE1 :=[diff(x(t),t)=y(t), diff(y(t),t)=-sin(x(t))];
DEplot( ODE1, [x(t),y(t)],t=-100..100,
[[x(0)=0,y(0)=a/8],[x(0)=0,y(0)=-a/2],
[x(0)=0.8*a,y(0)=0],[x(0)=a,y(0)=0]],
 x=-a..a,y=-a..a,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Phase portrait of the Nonlinear System",
arrows=SLIM);
```

$$ODE1 := \left[\frac{\partial}{\partial t}\mathrm{x}(t) = \mathrm{y}(t), \frac{\partial}{\partial t}\mathrm{y}(t) = -\sin(\mathrm{x}(t))\right]$$



4．.解:　1) $\dfrac{dx}{dt}+x-t=0$

$x=t-1+ce^{-t}$　　$x(0)=1$

$\phi(0)=c-1=1$　　$c=2$

$\phi(t)=t-1+2e^{-t}$　　$x(t)=t-1+(x_0+1)e^{-t}$

$\forall\varepsilon>0, \exists\delta=\varepsilon>0,$　　$\forall x(0)=x_0$

只要 $\|x_0-1\|<\delta$，就有

$$\|(x_0+1)e^{-t}+t-1-t+1-2e^{-t}\|=\|(x_0-1)e^{-t}\|<\|x_0-1\|<\delta=\varepsilon$$

$\therefore$　$x(t)=(t-1)+2e^{-t}$　是稳定的.

$\exists\delta_0>0$　(δ_0为常数),　$\forall\|x_0-1\|<\delta_0$

$$\lim_{t\to+\infty}\|(x_0-1)e^{-t}+t-1-t+1-2e^{-t}\|=\lim_{t\to+\infty}\|(x_0-1)e^{-t}\|<\lim_{t\to+\infty}e^{-t}\delta_0=0$$

$\therefore$ 解 $x=t-1+2e^{-t}$ 是渐近稳定的.

2) $\dfrac{dx}{dt}=\dfrac{x-x^3}{2t}$,　可解得特解为　$x(t)=0$　通解为：

$$x(t)=\pm x_0\sqrt{\frac{t}{1+x_0^2(t-1)}},\quad \text{对任意} x_0>0 \text{和} t>1, \text{有}$$

$$\lim_{t\to+\infty}|x(t)|=\lim_{t\to+\infty}\left|x_0\sqrt{\frac{t}{1+x_0^2(t-1)}}\right|\lim_{t\to+\infty}\left|\sqrt{\frac{1}{1+\dfrac{1-x_0^2}{x_0^2\,t}}}\right|=1.$$

解　　$x(t)=0$　是不稳定的

5．证明：1）$r>0$ 时，　$\frac{dr}{dt}=r^2\sin\frac{1}{r}$，　$\lim_{r\to 0}r^2\sin\frac{1}{r}=0$

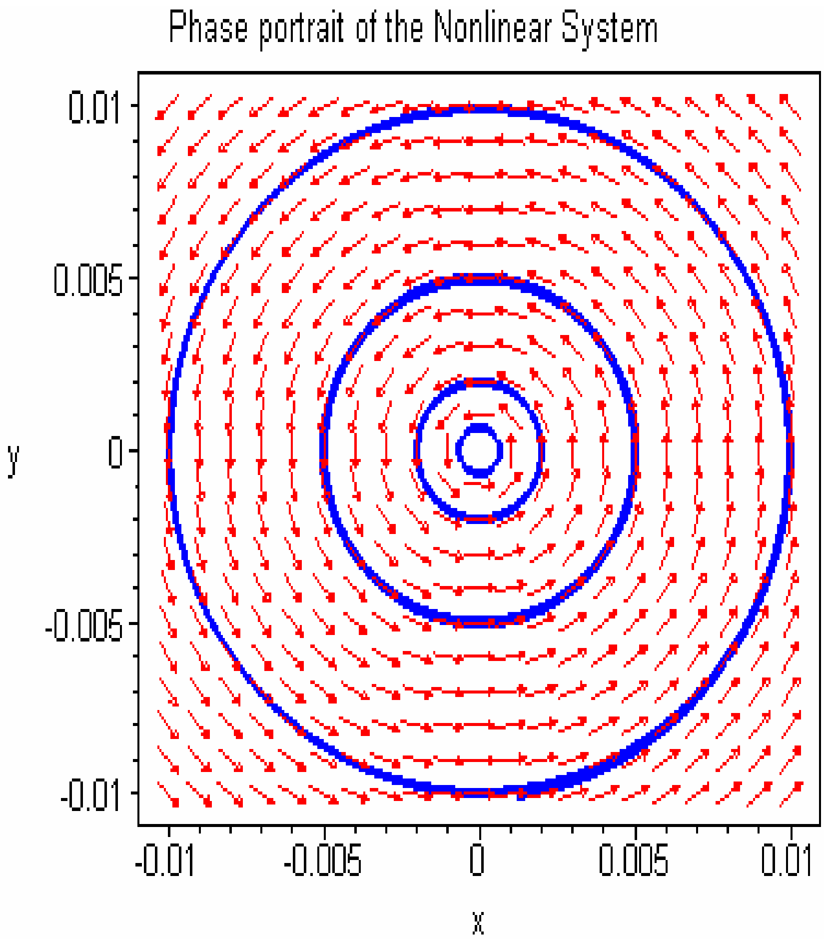
而 $r=0$ 时，　$\frac{dr}{dt}=0$

当　$r=\frac{1}{k\pi}$ 时，$\frac{dr}{dt}=0$，　$r=\frac{1}{k\pi}$，　$\theta=t+c$　是解，其余的是非闭合的轨线，都在某一格 $r=\frac{1}{k\pi}$ 内，平衡点(0 , 0)稳定.。

找到该系统对应的直角坐标方程，利用Maple画图。
注：　实际的轨线是有许多闭曲线和不闭曲线。

```
restart:with(DEtools): a:=1/100:
ODE1 :=[diff(x(t),t)=-y(t)+x(t)*sqrt(x(t)^2+y(t)^2)*sin(1
/sqrt(x(t)^2+y(t)^2)),
diff(y(t),t)=x(t)+y(t)*sqrt(x(t)^2+y(t)^2)*sin(1/sqrt(x(t
)^2+y(t)^2))];
DEplot( ODE1, [x(t),y(t)],t=-100..100,
[[x(0)=0,y(0)=a/15],[x(0)=0,y(0)=a/2],
[x(0)=0,y(0)=a/5],[x(0)=a,y(0)=0]],
 x=-a..a,y=-a..a,stepsize=0.05, dirgrid=[21,21],
 color=red, linecolor=blue,axes=BOXED,
 title="Phase portrait of the Nonlinear System",
arrows=SLIM);
```

$$ODE1:=\left[\frac{\partial}{\partial t}\mathrm{x}(t)=-\mathrm{y}(t)+\mathrm{x}(t)\sqrt{\mathrm{x}(t)^2+\mathrm{y}(t)^2}\,\sin\left(\frac{1}{\sqrt{\mathrm{x}(t)^2+\mathrm{y}(t)^2}}\right),\right.$$
$$\left.\frac{\partial}{\partial t}\mathrm{y}(t)=\mathrm{x}(t)+\mathrm{y}(t)\sqrt{\mathrm{x}(t)^2+\mathrm{y}(t)^2}\,\sin\left(\frac{1}{\sqrt{\mathrm{x}(t)^2+\mathrm{y}(t)^2}}\right)\right]$$



Phase portrait of the Nonlinear System
0.01
0.005
0
-0.005
-0.01
y
-0.01
-0.005
0
0.005
0.01
x