理论力学

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目录

1 质点运动学

1.1 运动的描述

1. 速度：

$$\boldsymbol{v}=\lim_{\Delta t\to 0}\frac{\Delta \boldsymbol{r}}{\Delta t}=\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}=\dot{\boldsymbol{r}} \tag{1.1}$$

2. 加速度：

$$\boldsymbol{a}=\lim_{\Delta t\to 0}\frac{\Delta \boldsymbol{v}}{\Delta t}=\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\dot{\boldsymbol{v}}=\ddot{\boldsymbol{r}} \tag{1.2}$$

1.2 速度和加速度的分量表示

1. 直角坐标系 (x,y,z)：

$$\boldsymbol{r}=x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z \tag{1.3}$$

$$\boldsymbol{v}=\dot{x}\boldsymbol{e}_x+\dot{y}\boldsymbol{e}_y+\dot{z}\boldsymbol{e}_j \tag{1.4}$$

$$\boldsymbol{a}=\ddot{x}\boldsymbol{e}_x+\ddot{y}\boldsymbol{e}_y+\ddot{z}\boldsymbol{e}_z \tag{1.5}$$

2. 柱坐标系 (ρ,φ,z)：

$$\boldsymbol{\rho}=\rho\boldsymbol{e}_\rho+z\boldsymbol{e}_z \tag{1.6}$$

$$\boldsymbol{v}=\dot{\rho}\boldsymbol{e}_\rho+\rho\dot{\varphi}\boldsymbol{e}_\varphi+\dot{z}\boldsymbol{e}_z \tag{1.7}$$

$$\boldsymbol{a}=(\ddot{\rho}-\rho\dot{\varphi}^2)\boldsymbol{e}_\rho+(\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi})\boldsymbol{e}_\varphi+\ddot{z}\boldsymbol{e}_z \tag{1.8}$$

3. 球坐标系 (r,θ,φ)：

$$\boldsymbol{r}=r\boldsymbol{e}_r \tag{1.9}$$

$$\boldsymbol{v}=\dot{r}\boldsymbol{e}_r+r\dot{\theta}\boldsymbol{e}_\theta+r\sin\theta\dot{\varphi}\boldsymbol{e}_\varphi \tag{1.10}$$

$$\begin{aligned}\boldsymbol{a}=&(\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta)\boldsymbol{e}_r+\\&(r\ddot{\theta}+2\dot{r}\dot{\theta}-r\dot{\varphi}^2\sin\theta\cos\theta)\boldsymbol{e}_\theta+\\&(r\ddot{\varphi}\sin\theta+2\dot{r}\dot{\varphi}\sin\theta+2r\dot{\varphi}\dot{\theta}\cos\theta)\boldsymbol{e}_\varphi\end{aligned} \tag{1.11}$$

4. 曲线坐标系 (q_1,q_2,q_3)：

$$\boldsymbol{v}=\boldsymbol{e}_1h_1\dot{q}_1+\boldsymbol{e}_2h_2\dot{q}_2+\boldsymbol{e}_3h_3\dot{q}_3 \tag{1.12}$$

$$a_i=\frac{1}{h_i}\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_i}\left(\frac{v^2}{2}\right)-\frac{\partial}{\partial q_i}\left(\frac{v^2}{2}\right)\right] \tag{1.13}$$

5. 自然坐标系 (τ,n,b)：

$$\boldsymbol{v}=v\boldsymbol{e}_\tau=\frac{\mathrm{d}s}{\mathrm{d}t}\boldsymbol{e}_\tau \tag{1.14}$$

$$\boldsymbol{a}=\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\boldsymbol{e}_\tau+v\frac{\mathrm{d}\boldsymbol{e}_\tau}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{e}_\tau+\frac{v^2}{\rho}\boldsymbol{e}_n \tag{1.15}$$

表 1: 常用曲率半径

曲线类型	曲率半径
$r=(x,y(x),0)$	$\rho=\dfrac{(1+y'^2)^{3/2}}{\lvert y''\rvert}$
$r=(x(t),y(t),0)$	$\rho=\dfrac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\lvert\dot{x}\ddot{y}-\dot{y}\ddot{x}\rvert}$
$r=(x(t),y(t),z(t))$	$\rho=(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)^{-1/2}$

2 质点动力学

2.1 Newton 运动定律

1. Newton 第一定律：物体保持匀速直线运动或静止状态，直到外力迫使它改变运动状态。

$$\boldsymbol{F}=\boldsymbol{0}\Rightarrow\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=\boldsymbol{0} \tag{2.1}$$

2. Newton 第二定律：物体的加速度与作用力成正比，与质量成反比，与作用力的方向相同。

$$\boldsymbol{F}=\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=m\boldsymbol{a} \tag{2.2}$$

3. Newton 第三定律：两物体间作用力和反作用力等大反向，作用在同一条直线上。

$$\boldsymbol{F}_{12}=-\boldsymbol{F}_{21} \tag{2.3}$$

2.2 质点动力学方程

$$m\ddot{\boldsymbol{r}}=\boldsymbol{F}(\boldsymbol{r},\dot{\boldsymbol{r}},t) \tag{2.4}$$

1. 直角坐标系 (x,y,z)：

$$\begin{cases}m\ddot{x}=F_x(x,y,z;\dot{x},\dot{y},\dot{z};t)\\ m\ddot{y}=F_y(x,y,z;\dot{x},\dot{y},\dot{z};t)\\ m\ddot{z}=F_z(x,y,z;\dot{x},\dot{y},\dot{z};t)\end{cases} \tag{2.5}$$

2. 柱坐标系 (ρ,φ,z)：

$$\begin{cases}m(\ddot{\rho}-\rho\dot{\varphi}^2)=F_\rho(\rho,\varphi,z;\dot{\rho},\dot{\varphi},\dot{z};t)\\ \dfrac{m}{\rho}\dfrac{\mathrm{d}}{\mathrm{d}t}(\rho\dot{\varphi})=F_\theta(\rho,\varphi,z;\dot{\rho},\dot{\varphi},\dot{z};t)\\ m\ddot{z}=F_\rho(\rho,\varphi,z;\dot{\rho},\dot{\varphi},\dot{z};t)\end{cases} \tag{2.6}$$

3. 球坐标系 (r,θ,φ)：

$$\begin{cases}m(\ddot{r}-r\dot{\theta}^2-r\dot{\varphi}^2\sin^2\theta)=F_r(r,\theta,\varphi;\dot{r},\dot{\theta},\dot{\varphi};t)\\ \dfrac{m}{r}\left[\dfrac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}^2)-r^2\dot{\varphi}^2\sin\theta\cos\theta\right]=F_\theta(r,\theta,\varphi;\dot{r},\dot{\theta},\dot{\varphi};t)\\ \dfrac{m}{r\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\varphi}\sin^2\theta)=F_\varphi(r,\theta,\varphi;\dot{r},\dot{\theta},\dot{\varphi};t)\end{cases} \tag{2.7}$$

4. 自然坐标系 (τ,n,b)：

$$\begin{cases}m\dot{v}=F_\tau(\tau,n,b;\dot{\tau},\dot{n},\dot{b};t)\\ m\dfrac{v^2}{\rho}=F_n(\tau,n,b;\dot{\tau},\dot{n},\dot{b};t)\\ 0=F_b(\tau,n,b;\dot{\tau},\dot{n},\dot{b};t)\end{cases} \tag{2.8}$$

2.3 质点的动量定理和动量守恒定律

1. 动量:

$$\boldsymbol{p} = m\boldsymbol{v} \tag{2.9}$$

2. 动量定理:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F} \tag{2.10}$$

$$\boldsymbol{p}_2 - \boldsymbol{p}_1 = \int_{t_1}^{t_2} \boldsymbol{F}\mathrm{d}t \tag{2.11}$$

3. 动量守恒定律:

$$\boldsymbol{F} = 0 \;\Rightarrow\; \boldsymbol{p} = \mathrm{Const} \tag{2.12}$$

2.4 质点的角动量定理和角动量守恒定律

1. 力矩:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} \tag{2.13}$$

2. 角动量:

$$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p} \tag{2.14}$$

3. 角动量定理:

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{M} \tag{2.15}$$

$$\boldsymbol{J}_2 - \boldsymbol{J}_1 = \int_{t_1}^{t_2} \boldsymbol{M}\mathrm{d}t \tag{2.16}$$

4. 角动量守恒定律:

$$\boldsymbol{M} = 0 \;\Rightarrow\; \boldsymbol{J} = \mathrm{Const} \tag{2.17}$$

2.5 质点的动能定理和机械能守恒定律

1. 功:

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \boldsymbol{v}\mathrm{d}t \tag{2.18}$$

2. 功率:

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \boldsymbol{F} \cdot \boldsymbol{v} \tag{2.19}$$

3. 动能:

$$T = \frac{1}{2}m\boldsymbol{v}^2 \tag{2.20}$$

4. 动能定理:

$$\mathrm{d}W = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}\left(\frac{1}{2}m\boldsymbol{v}^2\right) = \mathrm{d}T \tag{2.21}$$

$$W = \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \frac{1}{2}m\boldsymbol{v}_2^2 - \frac{1}{2}m\boldsymbol{v}_1^2 \tag{2.22}$$

5. 势能:

$$V = -\int_{\infty}^{\boldsymbol{r}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{2.23}$$

6. 保守力:

$$\oint_L \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0 \Leftrightarrow \nabla \times \boldsymbol{F} = 0 \tag{2.24}$$

$$\boldsymbol{F} = -\nabla V \tag{2.25}$$

7. 机械能守恒定律:保守力

$$T + V = E \tag{2.26}$$

3 质点系动力学

质心：

$$\boldsymbol{r}_c=\frac{\sum\limits_{i=1}^{n}m_i\boldsymbol{r}_i}{\sum\limits_{i=1}^{n}m_i},\quad \boldsymbol{v}_c=\frac{\sum\limits_{i=1}^{n}m_i\boldsymbol{v}_i}{\sum\limits_{i=1}^{n}m_i},\quad \boldsymbol{a}_c=\frac{\sum\limits_{i=1}^{n}m_i\boldsymbol{a}_i}{\sum\limits_{i=1}^{n}m_i},\quad m=\sum_{i=1}^{n}m_i \tag{3.1}$$

3.1 质点系的动量定理和动量守恒定律

1. 质点系的动量：

$$\boldsymbol{p}=\boldsymbol{p}_c+\boldsymbol{p}'=\boldsymbol{p}_c=m\boldsymbol{v}_c \tag{3.2}$$

2. 质点系的动量定理：

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}m_i\boldsymbol{v}_i=\sum_{i=1}^{n}\boldsymbol{F}_i^{(e)} \tag{3.3}$$

3. 质点系的动量守恒定律：

$$\sum_{i=1}^{n}\boldsymbol{F}_i^{(e)}=0\ \Rightarrow\ \boldsymbol{p}=\boldsymbol{p}_c=\mathrm{Const} \tag{3.4}$$

4. 质心运动定理：

$$m\ddot{r}_c=\sum_{i=1}^{n}\boldsymbol{F}_i \tag{3.5}$$

3.2 质点系的角动量定理和角动量守恒定律

1. 质点系的角动量：

$$\boldsymbol{J}=\boldsymbol{J}_c+\boldsymbol{J}'=\boldsymbol{r}_c\times\boldsymbol{p}_c+\sum_{i=1}^{n}\boldsymbol{r}_i'\times\boldsymbol{p}_i' \tag{3.6}$$

2. 质点系的角动量定理：

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n}\boldsymbol{r}_i\times m_i\boldsymbol{v}_i=\sum_{i=1}^{n}\boldsymbol{r}_i\times\boldsymbol{F}_i^{(e)}=\sum_{i=1}^{n}\boldsymbol{M}_i^{(e)} \tag{3.7}$$

3. 质点系的角动量守恒定律：

$$\sum_{i=1}^{n}\boldsymbol{M}_i^{(e)}=0\ \Rightarrow\ \boldsymbol{J}=\mathrm{Const} \tag{3.8}$$

3.3 质点系的动能定理和机械能守恒定律

1. Koenig 定理：

$$T=T_c+T'=\frac{1}{2}m\boldsymbol{v}_c^2+\frac{1}{2}\sum_{i=1}^{n}m_i\dot{\boldsymbol{r}}_i^2 \tag{3.9}$$

2. 质点系的动能定理：

$$\mathrm{d}T=\mathrm{d}\sum_{i=1}^{n}\frac{1}{2}m_i\boldsymbol{v}_i^2=\sum_{i=1}^{n}\boldsymbol{F}_i^{(e)}\cdot\mathrm{d}\boldsymbol{r}_i+\sum_{i=1}^{n}\boldsymbol{F}_i^{(i)}\cdot\mathrm{d}\boldsymbol{r}_i \tag{3.10}$$

3. 质点系的机械能守恒定律：外力和内力是保守力

$$T+V^{(e)}+U^{(i)}=E \tag{3.11}$$

3.4 两体问题

1. 折合质量:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{3.12}$$

2. 相对运动方程:

$$\mu \ddot{r}_{21} = \boldsymbol{F}_{21}^{(i)} + \frac{1}{m_1 + m_2}\left(m_1 \boldsymbol{F}_2^{(e)} - m_2 \boldsymbol{F}_1^{(e)}\right) \tag{3.13}$$

3. 相对质心的位矢:

$$\boldsymbol{r}_1' = -\frac{m_2}{m_1 + m_2}\boldsymbol{r}_{21}, \quad \boldsymbol{r}_2' = \frac{m_1}{m_1 + m_2}\boldsymbol{r}_{21} \tag{3.14}$$

4. 相对质心的角动量:

$$\boldsymbol{J}' = \boldsymbol{r}_1' \times m_i \boldsymbol{v}_i' + \boldsymbol{r}_2' \times m_2 \boldsymbol{v}_2' = \boldsymbol{r}_{21} \times \mu \boldsymbol{v}_{21} \tag{3.15}$$

5. 相对质心的动能:

$$T' = \frac{1}{2} m_1 \boldsymbol{v}_1'^2 + \frac{1}{2} m_2 \boldsymbol{v}_2'^2 = \frac{1}{2}\mu \boldsymbol{v}_{21}^2 \tag{3.16}$$

3.5 质心系和实验室系

1. 质心速度:

$$(m_1 + m_2)\boldsymbol{v}_c = m_1 \boldsymbol{v}_{1i} \tag{3.17}$$

2. 碰撞前速度:

$$\boldsymbol{v}_{1i}' = \boldsymbol{v}_{1i} - \boldsymbol{v}_c = \frac{m_2}{m_1 + m_2}\boldsymbol{v}_{1i} \tag{3.18}$$

$$\boldsymbol{v}_{2i}' = \boldsymbol{v}_{2i} - \boldsymbol{v}_c = -\frac{m_1}{m_1 + m_2}\boldsymbol{v}_{1i} \tag{3.19}$$

3. 能量守恒:

$$\frac{1}{2} m_1 \boldsymbol{v}_{1i}'^2 + \frac{1}{2} m_2 \boldsymbol{v}_{2i}'^2 = \frac{1}{2} m_1 \boldsymbol{v}_{1f}'^2 + \frac{1}{2} m_2 \boldsymbol{v}_{2f}'^2 \tag{3.20}$$

4. 碰撞后速度:

$$v_{1f}' = \frac{m_2}{m_1 + m_2} v_{1i}, \quad v_c = \frac{m_1}{m_1 + m_2} v_{1i} \tag{3.21}$$

5. 散射角关系:

$$\tan\theta_L = \frac{v_1' \sin\theta_c}{v_1' \cos\theta_c + v_c} = \frac{\sin\theta_c}{\cos\theta_c + m_1/m_2} \tag{3.22}$$

3.6 Virial 定理

1. 构造下述物理量(假定有限):

$$G = \sum_{i=1}^{n} \boldsymbol{r}_i \cdot \boldsymbol{p}_i \tag{3.23}$$

2. 取微商:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_{i=1}^{n} \dot{\boldsymbol{r}}_i \cdot \boldsymbol{p}_i + \sum_{i=1}^{n} \dot{\boldsymbol{p}}_i \cdot \boldsymbol{r}_i = 2T + \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \boldsymbol{r}_i \tag{3.24}$$

3. 取平均:

$$\langle 2T \rangle + \left\langle \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \boldsymbol{r}_i \right\rangle = \frac{1}{\tau}\int_0^{\tau} \frac{\mathrm{d}G}{\mathrm{d}t}\mathrm{d}t = \frac{1}{\tau}[G(\tau) - G(0)] = 0 \tag{3.25}$$

4. Virial 定理:

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \boldsymbol{r}_i \right\rangle \tag{3.26}$$

4 有心力

4.1 运动方程

1. 运动方程:

$$m\left(\ddot{r}-r\dot{\theta}^2\right)=F(r) \tag{4.1}$$

$$r^2\dot{\theta}=h \tag{4.2}$$

2. 变量代换:

$$\dot{\theta}=hu^2 \tag{4.3}$$

$$\dot{r}=\frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{u}\right)\dot{\theta}=-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta}=-h\frac{\mathrm{d}u}{\mathrm{d}\theta} \tag{4.4}$$

$$\ddot{r}=\frac{\mathrm{d}\dot{r}}{\mathrm{d}t}=-h\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)\dot{\theta}=-h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} \tag{4.5}$$

3. Binet 公式:

$$h^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)=-\frac{F(r)}{m},\quad u=\frac{1}{r} \tag{4.6}$$

4. 在平方反比引力作用下:

$$F(r)=-\alpha u^2 \tag{4.7}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=\frac{\alpha}{mh^2} \tag{4.8}$$

$$u=\frac{\alpha}{mh^2}+A\cos(\theta-\theta_0) \tag{4.9}$$

$$r=\frac{\dfrac{mh^2}{\alpha}}{1+\dfrac{Amh^2}{\alpha}\cos(\theta-\theta_0)} \tag{4.10}$$

4.2 守恒量和有效势

1. 能量和角动量守恒:

$$E=\frac{1}{2}\left(\dot{r}^2+r^2\dot{\theta}^2\right)+V(r)=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}+V(r) \tag{4.11}$$

$$J=mr^2\dot{\theta}=mh \tag{4.12}$$

2. 有效势:

$$V_{\text{eff}}(r)=\frac{J^2}{2mr^2}+V(r) \tag{4.13}$$

3. 在平方反比引力作用下:

$$V(r)=-\frac{\alpha}{r} \tag{4.14}$$

$$\dot{r}=\dot{\theta}\frac{\mathrm{d}r}{\mathrm{d}\theta}=\frac{J}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{4.15}$$

$$\mathrm{d}\theta=\frac{J\mathrm{d}r}{r^2\sqrt{2m\left(E-V(r)+\dfrac{J^2}{2mr^2}\right)}} \tag{4.16}$$

$$r=\frac{p}{1+e\cos(\theta-\theta_0)},\quad p=\frac{J^2}{m\alpha},\quad e=\sqrt{1+\frac{2EJ^2}{m\alpha^2}} \tag{4.17}$$

4. Lagrange-Runge-Lenz 矢量:

$$\boldsymbol{R}=\boldsymbol{p}\times\boldsymbol{J}-\frac{m\alpha\boldsymbol{r}}{r} \tag{4.18}$$

4.3 Kepler 定律

1. Kepler 第一定律：行星绕太阳作椭圆运动，太阳位于椭圆的一个焦点上。

$$r(\theta) = \frac{p}{1 + e\cos(\theta - \theta_0)} \tag{4.19}$$

2. Kepler 第二定律：行星和太阳之间的连线，在相等时间内所扫过的面积相等。

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r\dot{\theta}^2 = \frac{1}{2}h \tag{4.20}$$

3. Kepler 第三定律：行星公转周期的平方和轨道半长轴的立方成正比。

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M_s + M)} \approx \frac{4\pi^2}{GM_s} \tag{4.21}$$

4. 第一宇宙速度：

$$mg = m\frac{v_1^2}{R} \tag{4.22}$$

$$v = \sqrt{gR} = \sqrt{\frac{GM}{R}} = 7.9\ \mathrm{km/s} \tag{4.23}$$

5. 第二宇宙速度：

$$\frac{1}{2}mv_2^2 - G\frac{Mm}{R} = 0 \tag{4.24}$$

$$v_2 = \sqrt{\frac{2GM}{R}} = \sqrt{2}v_1 = 11.2\ \mathrm{km/s} \tag{4.25}$$

6. 第三宇宙速度：

$$\frac{1}{2}mv_3^2 = \frac{1}{2}mv_2^3 + \frac{1}{2}mv^2 \tag{4.26}$$

$$\frac{1}{2}mv'^2 - G\frac{M_s m}{d} = 0 \tag{4.27}$$

$$v = v' - \omega d \tag{4.28}$$

$$v_3 = 16.7\ \mathrm{km/s} \tag{4.29}$$

4.4 轨道闭合和稳定性问题

1. 轨道闭合条件：

$$\Delta\theta = \theta_2 - \theta_0 = 2\int_{r_{min}}^{r_{max}} \frac{J\mathrm{d}r}{r^2\sqrt{2m\left(E - V(r) + \frac{J^2}{2mr^2}\right)}} = \frac{m}{n}2\pi \tag{4.30}$$

2. Bertrand 定理：满足轨道闭合的势为以下之一：

$$V(r) = -\frac{\alpha}{r}, \quad V(r) = \frac{1}{2}\beta r^2 \tag{4.31}$$

3. 圆轨道稳定的条件：

$$\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}r} = 0, \quad \frac{\mathrm{d}^2V_{\mathrm{eff}}}{\mathrm{d}r^2} > 0 \tag{4.32}$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{J^2}{mr^3}, \quad 3\frac{\mathrm{d}V}{\mathrm{d}r} + r\frac{\mathrm{d}^2V}{\mathrm{d}r^2} > 0 \tag{4.33}$$

4.5 有心力场的散射

1. Coulomb 散射公式：

$$b(\theta) = \frac{|\alpha|}{mv_\infty^2}\cot\frac{\theta}{2} \tag{4.34}$$

2. Rutherford 散射公式：

$$\sigma_c(\theta) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\alpha}{2mv_\infty^2}\right)^2\frac{1}{\sin^4\frac{\theta}{2}} \tag{4.35}$$

5 刚体力学

5.1 刚体动力学方程

1. 质心运动定理:

$$m\ddot{\boldsymbol{r}}_c = \sum_{i=1}^{n} \boldsymbol{F}_i^{(e)} \tag{5.1}$$

2. 角动量定理:

$$\frac{\mathrm{d}\boldsymbol{J}'}{\mathrm{d}t} = \sum_{i=1}^{n} \boldsymbol{r}_i' \times \boldsymbol{F}_i^{(e)} = \sum_{i=1}^{n} \boldsymbol{M}_i' \tag{5.2}$$

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{J}_c + \boldsymbol{J}'\right) = \sum_{i=1}^{n} \left(\boldsymbol{r}_c + \boldsymbol{r}_i'\right) \times \boldsymbol{F}_i^{(e)} = \sum_{i=1}^{n} \boldsymbol{M}_i \tag{5.3}$$

3. 动能定理:

$$\mathrm{d}T = \sum_{i=1}^{n} \mathrm{d}\left(\boldsymbol{r}_c + \boldsymbol{r}_i'\right) \cdot \boldsymbol{F}_i^{(e)} = \sum_{i=1}^{n} \boldsymbol{r}_i \cdot \boldsymbol{F}_i^{(e)} \tag{5.4}$$

4. 刚体平衡条件:

$$\sum_{i=1}^{n} \boldsymbol{F}_i^{(e)} = 0, \quad \sum_{i=1}^{n} \boldsymbol{M}_i = 0 \text{ or } \sum_{i=1}^{n} \boldsymbol{M}_i' = 0 \tag{5.5}$$

5.2 Euler 角

1. Euler 角:

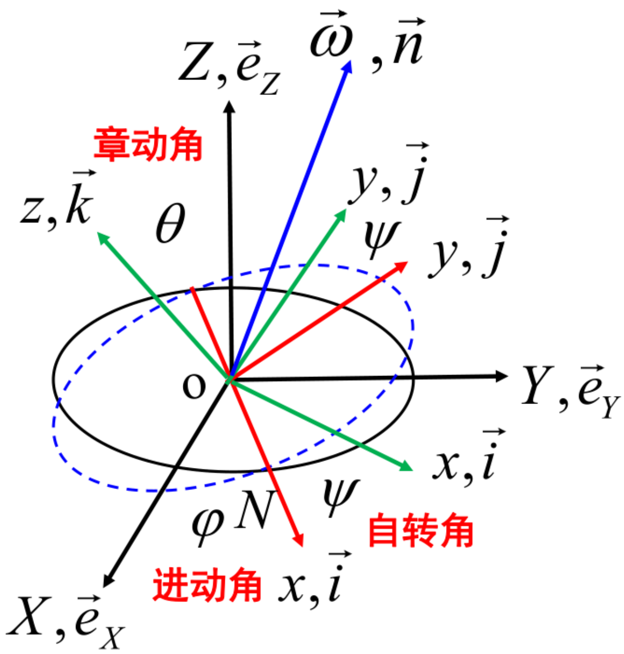

图 1: 刚体标架与 Euler 角

2. Euler 运动学方程:

$$\begin{cases} \omega_x = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_y = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \omega_z = \dot{\varphi}\cos\theta + \dot{\psi} \end{cases} \qquad \begin{cases} \omega_X = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \omega_Y = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi \\ \omega_Z = \dot{\psi}\cos\theta + \dot{\varphi} \end{cases} \tag{5.6}$$

5.3 惯量张量与转动惯量

5.3.1 刚体的角动量

1. 刚体的惯量张量:

$$\boldsymbol{I}=\int_V \rho(\boldsymbol{r})\mathrm{d}V\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} \tag{5.7}$$

$$I_{\alpha\beta}=\int_V \rho(\boldsymbol{r})\mathrm{d}V\left(\boldsymbol{r}^2\delta_{\alpha\beta}-x_\alpha x_\beta\right) \tag{5.8}$$

2. 刚体对 O 点的角动量:

$$\begin{aligned}\boldsymbol{J}&=\int_V \rho(\boldsymbol{r})\mathrm{d}V(\boldsymbol{r}\times\dot{\boldsymbol{r}})=\int_V \rho(\boldsymbol{r})\mathrm{d}V\left(\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})\right)\\ &=\int_V \rho(\boldsymbol{r})\mathrm{d}V\left(\boldsymbol{\omega}\boldsymbol{r}^2-(\boldsymbol{\omega}\cdot\boldsymbol{r})\boldsymbol{r}\right)=\boldsymbol{I}\boldsymbol{\omega}\end{aligned} \tag{5.9}$$

矩阵形式:

$$\boldsymbol{J}=\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{5.10}$$

$$J_\alpha=I_{\alpha\beta}\omega_\beta \tag{5.11}$$

5.3.2 刚体的转动动能

1. 刚体的转动动能利用惯量张量写为:

$$\begin{aligned}T&=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V\dot{\boldsymbol{r}}^2=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V(\boldsymbol{\omega}\times\boldsymbol{r})\cdot\dot{\boldsymbol{r}}=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V\left(\boldsymbol{r}\times\dot{\boldsymbol{r}}\right)\cdot\boldsymbol{\omega}\\ &=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{I}\boldsymbol{\omega}=\frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}\boldsymbol{\omega}=\frac{1}{2}\omega_\alpha I_{\alpha\beta}\omega_\beta\end{aligned} \tag{5.12}$$

矩阵形式:

$$T=\frac{1}{2}\begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix}\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{5.13}$$

2. 刚体的转动动能利用转动惯量写为:

$$\begin{aligned}T&=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V\dot{\boldsymbol{r}}^2=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V(\boldsymbol{\omega}\times\boldsymbol{r})^2\\ &=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V(\omega r\sin\theta)^2=\frac{1}{2}\int_V \rho(\boldsymbol{r})\mathrm{d}V\omega^2R^2=\frac{1}{2}I\omega^2\end{aligned} \tag{5.14}$$

3. 转动惯量和惯量张量的关系:

$$I=\boldsymbol{n}^T\boldsymbol{I}\boldsymbol{n} \tag{5.15}$$

矩阵形式:

$$I=\begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \end{pmatrix}\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \tag{5.16}$$

5.3.3 惯量椭球

1. 惯量椭球：
   (1) 选取矢量：

$$\boldsymbol{\rho}=\frac{\boldsymbol{n}}{\sqrt{I}}=\left(\frac{\cos\alpha}{\sqrt{I}},\frac{\cos\beta}{\sqrt{I}},\frac{\cos\gamma}{\sqrt{I}}\right)\quad,\quad I=\frac{1}{\boldsymbol{\rho}^2}\tag{5.17}$$

   (2) 式(5.15)可化为

$$\boldsymbol{\rho}^T\boldsymbol{I}\boldsymbol{\rho}=1\tag{5.18}$$

   (3) 展开得

$$I_{xx}x^2+I_{yy}y^2+I_{zz}z^2+2I_{xy}xy+2I_{xz}xz+2I_{yz}yz=1\tag{5.19}$$

2. 惯量主轴坐标系：惯量积 $I_{xy}=I_{xz}=I_{yz}=0$
   (1) 惯量椭球：

$$I_1x^2+I_2y^2+I_3z^2=1\tag{5.20}$$

   (2) 刚体的角动量：

$$\boldsymbol{J}=I_1\omega_x\boldsymbol{i}+I_2\omega_y\boldsymbol{j}+I_3\omega_z\boldsymbol{k}\tag{5.21}$$

   (3) 刚体的转动动能：

$$T=\frac{1}{2}\left(I_1\omega_x^2+I_2\omega_y^2+I_3\omega_z^2\right)\tag{5.22}$$

3. 求解惯量主轴坐标系：通过坐标系旋转（正交变换）消除惯量积

$$\boldsymbol{\rho}'=\boldsymbol{R}\boldsymbol{\rho}\Rightarrow\boldsymbol{\rho}=\boldsymbol{R}^T\boldsymbol{\rho}'\tag{5.23}$$

$$1=\boldsymbol{\rho}^T\boldsymbol{I}\boldsymbol{\rho}=\boldsymbol{\rho}'^T\boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^T\boldsymbol{\rho}'=\boldsymbol{\rho}'^T\boldsymbol{I}'\boldsymbol{\rho}'\tag{5.24}$$

5.4 刚体的平面平行运动

1. 刚体内一点的速度：

$$\boldsymbol{v}=\boldsymbol{v}_A+\boldsymbol{\omega}\times\boldsymbol{r}'\tag{5.25}$$

2. 刚体内一点的加速度：

$$\boldsymbol{a}=\boldsymbol{a}_A+\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{r}'-\omega^2\boldsymbol{r}'\tag{5.26}$$

3. 瞬时转轴：

$$x=\frac{v_{Ay}}{\omega},\quad y=-\frac{v_{Ax}}{\omega}\tag{5.27}$$

4. 角动量定理：

$$\frac{\mathrm{d}\boldsymbol{J}_z'}{\mathrm{d}t}=I_{zz}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}=\boldsymbol{M}_z'\tag{5.28}$$

$$\frac{\mathrm{d}\boldsymbol{J}_z'}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{J}_c+\boldsymbol{J}_z'\right)=\boldsymbol{M}_z\tag{5.29}$$

5. 能量守恒：

$$\frac{1}{2}mv_c^2+\frac{1}{2}I_{zz}\omega^2+V=E\tag{5.30}$$

5.5 刚体的定轴转动

1. 随动坐标系中的运动方程：

$$-mx_c\dot{\varphi}^2 - my_c\ddot{\varphi} = N_{Ax} + N_{Bx} + F_x \tag{5.31}$$

$$-my_c\dot{\varphi}^2 + mx_c\ddot{\varphi} = N_{Ay} + N_{By} + F_y \tag{5.32}$$

$$0 = N_{Az} + N_{Bz} + F_z \tag{5.33}$$

2. 角动量定理：

$$I_{xz}\ddot{\varphi} - I_{yz}\dot{\varphi}^2 = -N_{By}l_B + N_{Ay}l_A + M_{Fx} \tag{5.34}$$

$$I_{yz}\ddot{\varphi} + I_{xz}\dot{\varphi}^2 = N_{Bx}l_B - N_{Ax}l_A + M_{Fx} \tag{5.35}$$

$$I_{zz}\ddot{\varphi} = M_{Fz} \tag{5.36}$$

3. 能量：

$$T = \frac{1}{2}I_{zz}\dot{\varphi}^2, \quad E = \frac{1}{2}I_{zz}\dot{\varphi}^2 + V \tag{5.37}$$

4. 当 $\dot{\varphi} = 0$, $\ddot{\varphi} = 0$ 时，令 $l_A = 0$，

$$M_{Fx} - N_{By}l_B = 0 \tag{5.38}$$

$$M_{Fy} + N_{Bx}l_B = 0 \tag{5.39}$$

$$N_{Ax} + N_{Bx} + F_x = 0 \tag{5.40}$$

$$N_{Ay} + N_{By} + F_y = 0 \tag{5.41}$$

静反作用力：

$$N_{Bx}^s = -\frac{M_{Fy}}{l_B}, \quad N_{By}^s = \frac{M_{Fx}}{l_B} \tag{5.42}$$

$$N_{Ax}^s = -N_{Bx}^s - F_x, \quad N_{Ay}^s = -N_{By}^s - F_y \tag{5.43}$$

5. 当 $\dot{\varphi} \neq 0$ 时，令 $l_A = 0$，

$$I_{xz}\ddot{\varphi} - I_{yz}\dot{\varphi}^2 = -N_{By}l_B + M_{Fx} \tag{5.44}$$

$$I_{yz}\ddot{\varphi} + I_{xz}\dot{\varphi}^2 = N_{Bx}l_B + M_{Fx} \tag{5.45}$$

$$-mx_c\dot{\varphi}^2 - my_c\ddot{\varphi} = N_{Ax} + N_{Bx} + F_x \tag{5.46}$$

$$-my_c\dot{\varphi}^2 + mx_c\ddot{\varphi} = N_{Ay} + N_{By} + F_y \tag{5.47}$$

动反作用力：

$$N_{Bx}^d = -\frac{M_{Fy}}{l_B} + \frac{I_{xz}\dot{\varphi}^2 + I_{yz}\ddot{\varphi}}{l_B}, \quad N_{By}^d = \frac{M_{Fx}}{l_B} + \frac{I_{yz}\dot{\varphi}^2 - I_{xz}\ddot{\varphi}}{l_B} \tag{5.48}$$

$$N_{Ax}^d = -N_{Bx}^d - F_x - mx_c\dot{\varphi}^2 - my_c\ddot{\varphi}, \quad N_{Ay}^d = -N_{By}^d - F_y - my_c\dot{\varphi}^2 + mx_c\ddot{\varphi} \tag{5.49}$$

6. 无附加转动反作用力的充要条件：

$$I_{xz} = I_{yz} = 0, \quad x_c = y_c = 0 \tag{5.50}$$

5.6 刚体的定点转动

1. Euler 动力学方程:

$$\begin{cases} I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z = M_x \\ I_2\dot{\omega}_y - (I_3 - I_1)\omega_z\omega_x = M_y \\ I_3\dot{\omega}_z - (I_1 - I_2)\omega_x\omega_y = M_z \end{cases} \tag{5.51}$$

2. 能量守恒:

$$\frac{1}{2}(I_1\omega_x^2 + I_2\omega_y^2 + I_3\omega_z^2) + V = E \tag{5.52}$$

5.6.1 Lagrange-Possion 情况：重陀螺的定点转动

1. 主转动惯量:

$$I_{xx} = I_{yy} = I_1, \quad I_{zz} = I_3 \tag{5.53}$$

2. 三个守恒方程:

$$\frac{1}{2}(I_1\omega_x^2 + I_1\omega_y^2 + I_3\omega_z^2) + mgl\cos\theta = E \tag{5.54}$$

$$J_Z = \boldsymbol{J}\cdot\boldsymbol{e}_Z = I_1\omega_x\sin\theta\sin\psi + I_1\omega_y\sin\theta\cos\psi + I_3\omega_z\cos\theta = \alpha \tag{5.55}$$

$$J_z = I_3\omega_z = \beta \tag{5.56}$$

3. Euler 角:

$$\dot{\varphi} = \frac{J_Z - J_z\cos\theta}{I_1\sin^2\theta}, \quad \dot{\psi} = \frac{J_z}{I_3} - \frac{J_Z - J_z\cos\theta}{I_1\sin^2\theta}\cos\theta \tag{5.57}$$

$$\frac{1}{2}I_1\dot{\theta} + \frac{J_Z - J_z\cos\theta}{2I_1\sin^2\theta} + \frac{J_z^2}{2I_3} + mgl\cos\theta = E \;\Rightarrow\; \dot{\theta} \tag{5.58}$$

4. 高转速近似：$\dot{\psi} \gg \dot{\varphi}$

$$J_z = I_3\omega_z = I_3(\dot{\varphi}\cos\theta + \dot{\psi}) \approx I_3\dot{\psi} \tag{5.59}$$

$$J_Z = I_1\dot{\varphi}\sin^2\theta + J_z\cos\theta \approx J_z\cos\theta \tag{5.60}$$

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = I_3\dot{\psi}\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = I_3\dot{\psi}\boldsymbol{\omega}'\times\boldsymbol{k} = I_3\dot{\psi}\dot{\varphi}\boldsymbol{e}_Z\times\boldsymbol{k} = mgl(\cos\psi\boldsymbol{i} - \sin\psi\boldsymbol{j}) \tag{5.61}$$

$$\dot{\theta} = 0, \quad \dot{\varphi} = \frac{mgl}{I_3\dot{\psi}}, \quad \dot{\psi} = \frac{J_z}{I_3} \tag{5.62}$$

5.6.2 Euler-Pinsot 问题：自由刚体的定点转动

1. 主转动惯量:

$$I_{xx} = I_1, \quad I_{yy} = I_2, \quad I_{zz} = I_3 \tag{5.63}$$

2. 两个守恒方程:

$$\frac{1}{2}(I_1\omega_x^2 + I_2\omega_y^2 + I_3\omega_z^2) = E \tag{5.64}$$

$$I_1^2\omega_x^2 + I_2\omega_y^2 + I_3\omega_z^2 = J^2 \tag{5.65}$$

3. 用 ω_z 表示 ω_x 和 ω_y:

$$\omega_x = \pm\sqrt{\frac{J^2 - 2EI_2 - I_3(I_3 - I_2)\omega_z^2}{I_1(I_1 - I_2)}} = f_1(\omega_z) \tag{5.66}$$

$$\omega_y = \pm\sqrt{\frac{J^2 - 2EI_1 - I_3(I_3 - I_1)\omega_z^2}{I_2(I_2 - I_1)}} = f_2(\omega_z) \tag{5.67}$$

4. 求解 ω_z:

$$I_3\dot{\omega}_z - (I_1 - I_2)f_1(\omega_z)f_2(\omega_z) = 0 \Rightarrow \omega_z = \omega_z(t) \tag{5.68}$$

5.6.3 Euler-Pinsot 问题的特例

1. 主转动惯量：

$$I_{xx} = I_{yy} = I_1, \quad I_{zz} = I_3 \tag{5.69}$$

2. 求解 ω_x 和 ω_y：

$$\dot{\omega}_x = -\frac{I_3 - I_1}{I_1}\omega_z\omega_y = -n\omega_y \tag{5.70}$$

$$\dot{\omega}_y = \frac{I_3 - I_1}{I_1}\omega_z\omega_x = n\omega_x \tag{5.71}$$

$$\omega_x = \omega_0 \cos{(nt + \alpha)}, \quad \omega_y = \omega_0 \sin{(nt + \alpha)}, \quad \frac{I_3 - I_1}{I_1}\omega_z \tag{5.72}$$

3. 变化周期：

$$\tau = \frac{2\pi}{|n|} = \frac{2\pi}{\omega_z}\frac{I_1}{|I_3 - I_1|} \tag{5.73}$$

4. Euler 角：

$$\dot{\theta} = 0, \quad \dot{\varphi} = \frac{\omega_z + n}{\cos\theta}, \quad \dot{\psi} = -n \tag{5.74}$$

5. $\omega_X, \omega_Y, \omega_Z$：

$$\omega_X = -n\sin\theta\sin\left(\frac{\omega_z + n}{\cos\theta}t + \varphi_0\right) \tag{5.75}$$

$$\omega_Y = n\sin\theta\sin\left(\frac{\omega_z + n}{\cos\theta}t + \varphi_0\right) \tag{5.76}$$

$$\omega_Z = -n\cos\theta + \frac{\omega_z + n}{\cos\theta} \tag{5.77}$$

6 非惯性系力学

- 惯性系：Newton 运动定律成立的参考系。
- 非惯性系：相对于惯性系具有加速度 $\boldsymbol{a}_0$ 的参考系。
- 力学相对性原理：不能借助任何力学实验来判断惯性参考系是静止还是作匀速直线运动。
- 惯性力：为了使 Newton 第二定律形式上在非惯性系中仍成立而定义的虚拟力。

6.1 非惯性系运动学

1. 平动参考系：

(1) 绝对速度 = 相对速度 + 牵连速度

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}' \tag{6.1}$$

(2) 绝对加速度 = 相对加速度 + 牵连加速度

$$\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{a}' \tag{6.2}$$

2. 转动参考系：

(1) 绝对速度 = 相对速度 + 牵连速度

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}' \tag{6.3}$$

(2) 绝对加速度 = 相对加速度 + 牵连加速度 + Coriolis 加速度

$$\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{a}' + \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') + 2\boldsymbol{\omega} \times \boldsymbol{v}' \tag{6.4}$$

当 $\boldsymbol{\omega}$ 是常矢量时，

$$\boldsymbol{a} = \boldsymbol{a}' - \boldsymbol{\omega}^2\boldsymbol{R} + 2\boldsymbol{\omega} \times \boldsymbol{v}' \tag{6.5}$$

6.2 非惯性系动力学

1. 平动参考系：

$$m\boldsymbol{a}' = \boldsymbol{F} - m\boldsymbol{a}_0 \tag{6.6}$$

2. 转动参考系：

$$m\boldsymbol{a}' = \boldsymbol{F} - m\boldsymbol{a}_0 - m\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') - 2m\boldsymbol{\omega} \times \boldsymbol{v}' \tag{6.7}$$

当 $\boldsymbol{\omega}$ 是常矢量时，

$$m\boldsymbol{a}' = \boldsymbol{F} - m\boldsymbol{a}_0 + m\boldsymbol{\omega}^2\boldsymbol{R} - 2m\boldsymbol{\omega} \times \boldsymbol{v}' \tag{6.8}$$

6.3 地球自转的影响

1. 动力学方程：

$$m\boldsymbol{a}' = \boldsymbol{F} - mg\boldsymbol{k} - 2m\boldsymbol{\omega} \times \boldsymbol{v}' \tag{6.9}$$

2. 计算 $\boldsymbol{\omega} \times \boldsymbol{v}'$：

$$\begin{aligned}\boldsymbol{\omega} \times \boldsymbol{v}' &= (-\omega\cos\lambda\boldsymbol{i} + \omega\sin\lambda\boldsymbol{k}) \times (\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}) \\ &= -\omega\dot{y}\sin\lambda\boldsymbol{i} + (\omega\dot{x}\sin\lambda + \omega\dot{z}\cos\lambda)\boldsymbol{j} - \omega\dot{y}\cos\lambda\boldsymbol{k}\end{aligned} \tag{6.10}$$

3. 动力学分量方程：

$$\begin{cases} m\ddot{x} = F_x + 2m\omega\dot{y}\sin\lambda \\ m\ddot{y} = F_y - 2m\omega(\dot{x}\sin\theta + \dot{z}\cos\lambda) \\ m\ddot{z} = F_z - mg + 2m\omega\dot{y}\cos\lambda \end{cases} \tag{6.11}$$

4. 自由落体：

(1) 积分一次：

$$\begin{cases} \dot{x} = 2\omega y\sin\lambda \\ \dot{y} = -2\omega[x\sin\lambda + (z-h)\cos\lambda] \\ \dot{z} = -gt + 2\omega y\cos\lambda \end{cases} \tag{6.12}$$

(2) 代回动力学分量方程，并忽略 ω^2 项：

$$\begin{cases} \ddot{x} = 0 \\ \ddot{y} = 2gt\omega\cos\lambda \\ \ddot{z} = -g \end{cases} \tag{6.13}$$

(3) 运动学分量方程：

$$\begin{cases} x = 0 \\ y = \dfrac{1}{3}gt^3\omega\cos\lambda \\ z = h - \dfrac{1}{2}gt^2 \end{cases} \tag{6.14}$$

(4) 运动轨道方程：

$$y^2 = -\frac{8}{9}\frac{\omega^2\cos^2\lambda}{g}(z-h)^3 \tag{6.15}$$

(5) 落点偏东量：

$$y = \frac{1}{3}\sqrt{\frac{8h^3}{g}}\omega\cos\lambda \tag{6.16}$$

7 约束和广义坐标

1. 自由度 (p.d.f, physical degrees of freedom)：

$$s = 3N - k \tag{7.1}$$

2. 几何约束，又称完整约束 (holonomic constraint)：

$$f_i(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N; t) = 0, \quad i = 1, 2, \ldots, k \tag{7.2}$$

3. 运动约束，又称微分约束：有时经过积分后可变为几何约束，否则称为非完整约束

$$g(\xi_1, \xi_2, \ldots, \xi_M; \dot{\xi}_1, \dot{\xi}_2, \ldots, \dot{\xi}_M; t) = 0 \tag{7.3}$$

4. 广义坐标：

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \ldots, q_s, t), \quad i = 1, 2, \ldots, N \tag{7.4}$$

8 虚功原理

1. 理想约束 (ideal constraint)：

$$\sum_{i=1}^{n} \boldsymbol{F}_i^{(r)} \cdot \delta \boldsymbol{r}_i = 0 \tag{8.1}$$

2. 虚功原理：

$$\sum_{i=1}^{n} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = 0 \tag{8.2}$$

3. d'Alembert 原理：

$$\sum_{i=1}^{n} (\dot{\boldsymbol{p}}_i - \boldsymbol{F}_i) \cdot \delta \boldsymbol{r}_i = 0 \tag{8.3}$$

9 Lagrange 方程

9.1 第二类 Lagrange 方程

1. 虚位移：

$$\delta \boldsymbol{r}_i = \sum_{\alpha=1}^{s} \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \delta q_\alpha \tag{9.1}$$

2. 广义力：

$$Q_\alpha = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} \tag{9.2}$$

3. 用广义坐标写出 d'Alembert 原理：

$$\sum_{\alpha=1}^{s} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_\alpha} \right) - \frac{\partial T}{\partial q_\alpha} - Q_\alpha \right] \delta q_\alpha = 0 \tag{9.3}$$

4. 第二类 Lagrange 方程：

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_\alpha} \right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha, \quad \alpha = 1, \ldots, s \tag{9.4}$$

9.2 第一类 Lagrange 方程

1. 主动力是保守力：

$$\boldsymbol{F}_i = -\nabla V_i = -\left(\frac{\partial V}{\partial x_i}\boldsymbol{e}_x + \frac{\partial V}{\partial y_i}\boldsymbol{e}_y + \frac{\partial V}{\partial z_i}\boldsymbol{e}_z\right) \tag{9.5}$$

2. 广义力：

$$Q_\alpha = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} = -\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_\alpha} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_\alpha} + \frac{\partial V}{\partial z_1}\frac{\partial z_i}{\partial q_\alpha}\right) = -\frac{\partial V}{\partial q_\alpha} \tag{9.6}$$

3. Lagrange 量：

$$\mathcal{L} = T - V = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{\boldsymbol{r}}_i^2 - V(q_1,\ldots,q_s) \tag{9.7}$$

4. 第一类 Lagrange 方程：

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0, \quad \alpha = 1,\ldots,s \tag{9.8}$$

9.3 广义势

1. 广义势：依赖广义速度

$$U(q_1,\ldots,q_s;\dot{q}_1,\ldots,\dot{q}_s;t) \tag{9.9}$$

2. 广义力：

$$Q_\alpha = -\frac{\partial U}{\partial q_\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}_\alpha}\right) \tag{9.10}$$

3. Lagrange 方程：

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0, \quad \mathcal{L} = T - U \tag{9.11}$$

4. 正则动量 (canonical momentum)：

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}, \quad \alpha = 1,\ldots,s \tag{9.12}$$

9.4 耗散系统

1. 广义力：Q'_α 是无法用广义势函数表示的部分

$$Q_\alpha = -\frac{\partial U}{\partial q_\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{q}_\alpha}\right) + Q'_\alpha \tag{9.13}$$

2. Lagrange 方程：

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = Q'_\alpha, \quad \mathcal{L} = T - U \tag{9.14}$$

3. 流体中粒子的运动：
   (1) 耗散力：

$$F'_{(i),x} = -k_{(i),x}v_{(i),x}, \quad F'_{(i),y} = -k_{(i),y}v_{(i),y}, \quad F'_{(i),z} = -k_{(i),z}v_{(i),z} \tag{9.15}$$

   (2) Rayleigh 耗散函数：

$$\mathcal{F} = \frac{1}{2}\sum_{i=1}^{N}\left(k_{(i),x}v_{(i),x}^2 + k_{(i),y}v_{(i),y}^2 + k_{(i),z}v_{(i),z}^2\right) \tag{9.16}$$

   (3) 广义耗散力：

$$Q'_\alpha = \sum_{i=1}^{N}\boldsymbol{F}'_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_\alpha} = \sum_{i=1}^{N}\boldsymbol{F}'_i \cdot \frac{\partial \boldsymbol{v}_i}{\partial \dot{q}_\alpha} = -\frac{\partial \mathcal{F}}{\partial \dot{q}_\alpha} \tag{9.17}$$

   (4) Lagrange 方程：

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} + \frac{\partial \mathcal{F}}{\partial \dot{q}_\alpha} = 0, \quad \mathcal{L} = T - U \tag{9.18}$$

10 最小作用量原理

10.1 Euler 方程

1. 泛函：

$$J[\alpha, y] = \int_{x_1}^{x_2} f\left(y(\alpha, x), y'(\alpha, x); x\right) \mathrm{d}x \tag{10.1}$$

$$y(\alpha, x) = y(0, x) + \alpha\eta(x), \quad \eta(x_1) = \eta(x_2) = 0 \tag{10.2}$$

2. 变分记号：

$$\delta J = \frac{\partial J}{\partial \alpha}\mathrm{d}\alpha, \quad \delta y = \frac{\partial y}{\partial \alpha}\mathrm{d}\alpha, \quad \delta y' = \frac{\mathrm{d}}{\mathrm{d}x}\delta y \tag{10.3}$$

3. 变分原理：

$$\delta J = \delta \int_{x_1}^{x_2} f \mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right) \mathrm{d}x = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'}\right) \delta y \mathrm{d}x = 0 \tag{10.4}$$

4. Euler 方程：

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0 \tag{10.5}$$

5. Euler 方程第二形式：

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \tag{10.6}$$

6. 多函数的 Euler 方程：

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} = 0 \tag{10.7}$$

10.2 约束下的 Euler 方程

1. 代数约束下的 Euler 方程：

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j \frac{\partial g_j}{\partial y_i} = 0, \quad g_j(y_i; x) = 0 \tag{10.8}$$

2. 微分约束下的 Euler 方程：

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j \frac{\partial g_j}{\partial y_i} = 0, \quad \sum_i \frac{\partial g_i}{\partial y_i}\mathrm{d}y_i = 0 \tag{10.9}$$

3. 积分约束下的 Euler 方程：

$$\frac{\partial f}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y_i'} + \sum_j \lambda_j \left(\frac{\partial g_j}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g_j}{\partial y_i'}\right) = 0, \quad K[y] = \int_{x_1}^{x_2} g(y_i, y_i'; x)\mathrm{d}x = \ell \tag{10.10}$$

10.3 最小作用量原理

1. 分析力学中的泛函：

$$S[q] = \int_{t_1}^{t_2} \mathcal{L}(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s; t)\mathrm{d}t \tag{10.11}$$

2. 最小作用量原理或 Hamilton 原理：

$$\delta S[q] = \int_{t_1}^{t_2} \sum_{\alpha=1}^{s} \left[\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\right] \delta q_\alpha \mathrm{d}t = 0 \tag{10.12}$$

3. Euler-Lagrange 方程：

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) = 0, \quad \alpha = 1, \ldots, s \tag{10.13}$$

4. Lagrange 量的规范变换：

$$\widetilde{\mathcal{L}}(q, \dot{q}, t) = \mathcal{L}(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t}f(q, t) \tag{10.14}$$

10.4 非完整约束下的最小作用量原理

1. 非完整约束方程:

$$\sum_{\alpha=1}^{s} C_{l\alpha}\dot{q}_\alpha + D_l = 0, \quad l = 1, \ldots, p \tag{10.15}$$

2. 非完整约束下的 Euler-Lagrange 方程:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = \sum_{l=1}^{p} \lambda_l C_{l\alpha}, \quad \alpha = 1, \ldots, s \tag{10.16}$$

3. 约束力:

$$Q'_\alpha = \sum_{l=1}^{p} \lambda_l C_{l\alpha}, \quad \alpha = 1, \ldots, s \tag{10.17}$$

4. 完整约束下求解约束力:

$$f_l(q,t) = 0 \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{\alpha=1}^{s} \frac{\partial f_l}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f_l}{\partial t} = 0, \quad l = 1, \ldots, p \tag{10.18}$$

$$C_{l\alpha} = \frac{\partial f_l}{\partial q_\alpha}, \quad D_l = \frac{\partial f_l}{\partial t} \tag{10.19}$$

5. 完整约束下引入 Lagrange 乘子坐标:

$$\xi_1 = q_1, \ldots, \xi_s = q_s, \xi_{s+1} = \lambda_1, \ldots, \xi_{s+p} = \lambda_p \tag{10.20}$$

$$\widetilde{\mathcal{L}}(\xi, \dot{\xi}, t) = \mathcal{L}(q, \dot{q}, t) + \sum_{l=1}^{p} \lambda_l f_l(q,t) \tag{10.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{\xi}_\alpha}\right) - \frac{\partial \widetilde{\mathcal{L}}}{\partial \xi_\alpha} = 0, \quad \alpha = 1, \ldots, s+p \tag{10.22}$$

11 对称性与守恒量

11.1 守恒量与循环坐标

1. 守恒量或运动常数:

$$\frac{\mathrm{d}}{\mathrm{d}t}A(q,\dot{q};t) = \sum_{\alpha=1}^{s}\left(\frac{\partial A}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial A}{\partial \dot{q}_\alpha}\ddot{q}_\alpha\right) + \frac{\partial A}{\partial t} = 0 \tag{11.1}$$

2. 循环坐标或可遗坐标:Lagrange 量不依赖于坐标 q_α

$$\frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \tag{11.2}$$

3. 循环坐标的正则动量守恒:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial q_\alpha} = 0 \Rightarrow \frac{\mathrm{d}p_\alpha}{\mathrm{d}t} = 0 \tag{11.3}$$

4. 能量函数或 Jacobi 积分:

$$h = \sum_{\alpha=1}^{s} \dot{q}_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} - \mathcal{L} \tag{11.4}$$

5. 能量函数等于总能量的条件:*Cartesian* 坐标与广义坐标间的坐标变换不显含时间

$$T = M_0 + \sum_{\alpha=1}^{s} M_\alpha \dot{q}_\alpha + \sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} M_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta = \sum_{\alpha=1}^{s}\sum_{\beta=1}^{s} M_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta \tag{11.5}$$

$$h = \sum_{\alpha} \frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha - \mathcal{L} = 2T - \mathcal{L} = T + V = E \tag{11.6}$$

11.2 无穷小坐标变换与守恒定理

1. 动量守恒：空间平移对称性

$$\delta\mathcal{L}=\sum_{i=1}^{N}\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_i}\cdot\delta\boldsymbol{r}_i=\sum_{i=1}^{N}\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_i}\cdot(\varepsilon\boldsymbol{e}_n)=\sum_{i=1}^{N}\frac{\mathrm{d}\boldsymbol{p}_i}{\mathrm{d}t}\cdot(\varepsilon\boldsymbol{e}_n)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_n\cdot\sum_{i=1}^{N}\boldsymbol{p}_i\right)\varepsilon=\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_n\cdot\boldsymbol{P})\,\varepsilon=0 \tag{11.7}$$

2. 角动量守恒：空间转动对称性

$$\begin{aligned}\delta\mathcal{L}&=\sum_{i=1}^{N}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_i}\cdot\delta\boldsymbol{r}_i+\frac{\partial\mathcal{L}}{\partial v_i}\cdot\delta\boldsymbol{v}_i\right)=\sum_{i=1}^{N}\left(\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_i}\cdot(\delta\theta\boldsymbol{e}_n\times\boldsymbol{r}_i)+\frac{\partial\mathcal{L}}{\partial v_i}\cdot(\delta\theta\boldsymbol{e}_n\times\boldsymbol{v}_i)\right)\\&=\boldsymbol{e}_n\cdot\sum_{i=1}^{N}\left(\boldsymbol{r}_i\times\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}_i}+\boldsymbol{v}_i\times\frac{\partial\mathcal{L}}{\partial\boldsymbol{v}_i}\right)\delta\theta=\boldsymbol{e}_n\cdot\sum_{i=1}^{N}(\boldsymbol{r}_i\times\dot{\boldsymbol{p}}_i+\dot{\boldsymbol{r}}_i\times\boldsymbol{p}_i)\,\delta\theta\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_n\cdot\sum_{i=1}^{N}\boldsymbol{r}_i\times\boldsymbol{p}_i\right)\delta\theta=\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_n\cdot\boldsymbol{J})\,\delta\theta=0\end{aligned} \tag{11.8}$$

3. 能量函数守恒：时间平移对称性

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\sum_{\alpha=1}^{s}\left[\ddot{q}_\alpha\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}+\dot{q}_\alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}\right)\right]-\left[\sum_{\alpha=1}^{s}\left(\frac{\partial\mathcal{L}}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}\ddot{q}_\alpha\right)+\frac{\partial\mathcal{L}}{\partial t}\right]=-\frac{\partial\mathcal{L}}{\partial t}=0 \tag{11.9}$$

11.3 Noether 定理

1. 无穷小变换：

$$t'=t+\delta t=t+\varepsilon X(q,t) \tag{11.10}$$

$$q'_\alpha(t')=q_\alpha(t)+\delta q_\alpha(t)=q_\alpha(t)+\varepsilon\Psi_\alpha(q,t) \tag{11.11}$$

2. 连续对称：

$$\Delta S=\int_{t'_1}^{t'_2}\mathcal{L}\left(q',\frac{\mathrm{d}q'}{\mathrm{d}t'},t'\right)\mathrm{d}t'-\int_{t_1}^{t_2}\mathcal{L}\left(q,\frac{\mathrm{d}q}{\mathrm{d}t},t\right)\mathrm{d}t=0 \tag{11.12}$$

3. Noether 条件：

$$\sum_{\alpha=1}^{s}\left[\Psi_\alpha\frac{\partial\mathcal{L}}{\partial q_\alpha}+\left(\dot{\Psi}_\alpha-\dot{q}_\alpha\dot{X}\right)\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}\right]+\mathcal{L}\dot{X}+\frac{\partial\mathcal{L}}{\partial t}X=0 \tag{11.13}$$

4. Noether 定理：

$$C=\sum_{\alpha=1}^{s}\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}(\dot{q}_\alpha X-\Psi_\alpha)-X\mathcal{L} \tag{11.14}$$

11.4 Noether 定理的推广

1. 连续对称的推广：

$$\Delta S=\int_{t'_1}^{t'_2}\mathcal{L}\left(q',\frac{\mathrm{d}q'}{\mathrm{d}t'},t'\right)\mathrm{d}t'-\int_{t_1}^{t_2}\left[\mathcal{L}\left(q,\frac{\mathrm{d}q}{\mathrm{d}t},t\right)+\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}G(q,t)\right]\mathrm{d}t=0 \tag{11.15}$$

2. Noether 条件的推广：

$$\sum_{\alpha=1}^{s}\left[\Psi_\alpha\frac{\partial\mathcal{L}}{\partial q_\alpha}+\left(\dot{\Psi}_\alpha-\dot{q}_\alpha\dot{X}\right)\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}\right]+\mathcal{L}\dot{X}+\frac{\partial\mathcal{L}}{\partial t}X=\dot{G} \tag{11.16}$$

3. Noether 定理的推广：

$$\widetilde{C}=\sum_{\alpha=1}^{s}\frac{\partial\mathcal{L}}{\partial\dot{q}_\alpha}(\dot{q}_\alpha X-\Psi_\alpha)-X\mathcal{L}+G \tag{11.17}$$

12 小振动

12.1 简正模式

1. 动能是广义速度的二次型：

$$T=\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\left(\sum_{i=1}^{N}m_i\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\frac{\partial\boldsymbol{r}_i}{\partial q_\beta}\right)\dot{q}_\alpha\dot{q}_\beta=\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}M_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta \tag{12.1}$$

2. 势函数在平衡点处展开：

$$V(q_1,\ldots,q_s)=V_0+\sum_{\alpha=1}^{s}\left.\frac{\partial V}{\partial q_\alpha}\right|_{q=0}q_\alpha+\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\left(\left.\frac{\partial^2 V}{\partial q_\alpha\partial q_\beta}\right|_{q=0}\right)q_\alpha q_\beta+\mathcal{O}(q^3) \tag{12.2}$$

平衡点处第二项为 0，不妨令第一项 $V_0=0$，则势函数变为

$$V=\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}K_{\alpha\beta}q_\alpha q_\beta \tag{12.3}$$

3. 系统的 Lagrange 量：

$$\mathcal{L}=\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}M_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta-\frac{1}{2}\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}K_{\alpha\beta}q_\alpha q_\beta=\frac{1}{2}\left(\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}-\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}\right) \tag{12.4}$$

4. 小振动方程：

$$\sum_{\beta=1}^{s}\left(K_{\alpha\beta}q_\beta+M_{\alpha\beta}\ddot{q}_\beta\right)=0 \tag{12.5}$$

5. 试探解：

$$\boldsymbol{q}=\boldsymbol{D}\mathrm{e}^{-\mathrm{i}\omega t}+\boldsymbol{D}^*\mathrm{e}^{\mathrm{i}\omega t} \tag{12.6}$$

6. 代入小振动方程得到齐次线性方程组：

$$\sum_{\beta=1}^{s}\left(K_{\alpha\beta}-\omega^2M_{\alpha\beta}\right)D_\beta=0\Leftrightarrow\left(\boldsymbol{K}-\omega^2\boldsymbol{M}\right)\boldsymbol{D}=0 \tag{12.7}$$

7. 久期方程：

$$|\boldsymbol{K}-\omega^2\boldsymbol{M}|=0 \tag{12.8}$$

8. 系统的简正模式：

$$\boldsymbol{q}_n=\boldsymbol{D}_n\mathrm{e}^{-\mathrm{i}\omega_n t}+\boldsymbol{D}_n^*\mathrm{e}^{\mathrm{i}\omega_n t},\quad n=1,\ldots,s \tag{12.9}$$

9. 系统的通解：

$$\boldsymbol{q}=\sum_{n=1}^{s}\left(C_n\mathrm{e}^{-\mathrm{i}\phi_n}\boldsymbol{D}_n\mathrm{e}^{-\mathrm{i}\omega_n t}+C_n\mathrm{e}^{\mathrm{i}\phi_n}\boldsymbol{D}_n^*\mathrm{e}^{\mathrm{i}\omega_n t}\right) \tag{12.10}$$

10. 系统的物理解：取实部

$$\mathrm{Re}(\boldsymbol{q})=\sum_{n=1}^{s}C_n\boldsymbol{D}_n\cos\left(\omega_n t+\phi_n\right) \tag{12.11}$$

12.2 简正坐标

1. 简正坐标：

$$\boldsymbol{q}=\boldsymbol{\mathcal{C}}\boldsymbol{\eta}\Rightarrow\boldsymbol{\eta}=\boldsymbol{\mathcal{C}}^{-1}\boldsymbol{q}=\boldsymbol{\mathcal{C}}^T\boldsymbol{M}\boldsymbol{q} \tag{12.12}$$

2. 系统的 Lagrange 量：

$$\mathcal{L}=\frac{1}{2}\left(\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}}-\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q}\right)=\frac{1}{2}\left(\dot{\boldsymbol{\eta}}^T\boldsymbol{\mathcal{C}}^T\boldsymbol{M}\boldsymbol{\mathcal{C}}\dot{\boldsymbol{\eta}}-\boldsymbol{\eta}^T\boldsymbol{\mathcal{C}}^T\boldsymbol{K}\boldsymbol{\mathcal{C}}\boldsymbol{\eta}\right)=\frac{1}{2}\left(\dot{\boldsymbol{\eta}}^T\dot{\boldsymbol{\eta}}-\boldsymbol{\eta}^T\boldsymbol{\Lambda}\boldsymbol{\eta}\right) \tag{12.13}$$

13 Hamilton 正则方程

13.1 Hamilton 正则方程

1. Hamilton 量 (Hamiltonian)：

$$\mathcal{H}(q,p,t)=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-\mathcal{L}(q,\dot{q},t) \tag{13.1}$$

2. Hamilton 量的全微分：

$$\mathrm{d}\mathcal{H}=\sum_{\alpha=1}^{s}\left(\dot{q}_\alpha\mathrm{d}p_\alpha-\frac{\partial\mathcal{L}}{\partial q_\alpha}\mathrm{d}q_\alpha\right)-\frac{\partial\mathcal{L}}{\partial t}\mathrm{d}t=\sum_{\alpha=1}^{s}\left(\frac{\partial\mathcal{H}}{\partial q_\alpha}\mathrm{d}q_\alpha+\frac{\partial\mathcal{H}}{\partial q_\alpha}\mathrm{d}q_\alpha\right)+\frac{\partial\mathcal{H}}{\partial t}\mathrm{d}t \tag{13.2}$$

3. Hamilton 正则方程：

$$\dot{q}_\alpha=\frac{\partial\mathcal{H}}{\partial p_\alpha},\quad \dot{p}_\alpha=-\frac{\partial\mathcal{H}}{\partial q_\alpha} \tag{13.3}$$

4. Hamilton 量不含时意味着能量守恒：

$$\mathcal{H}(q,p)=\sum_{\alpha=1}^{s}p_\alpha\dot{q}_\alpha-\mathcal{L}(q,\dot{q})=E \tag{13.4}$$

5. Liouville 定理：相空间中的任意区域在时间演化过程中体积保持不变。
   (1) 广义坐标和正则动量的时间演化：

$$\begin{cases}\widetilde{q}_\alpha=q_\alpha+\dot{q}_\alpha\mathrm{d}t=q_\alpha+\dfrac{\partial\mathcal{H}}{\partial p_\alpha}\mathrm{d}t\\ \widetilde{p}_\alpha=p_\alpha+\dot{q}_\alpha\mathrm{d}t=p_\alpha-\dfrac{\partial\mathcal{H}}{\partial q_\alpha}\mathrm{d}t\end{cases} \tag{13.5}$$

   (2) 相空间体积元的时间演化：

$$\mathrm{d}\widetilde{\Omega}=\prod_\alpha\mathrm{d}\widetilde{q}_\alpha\mathrm{d}\widetilde{p}_\alpha=|\boldsymbol{\mathcal{J}}|\prod_\alpha\mathrm{d}q_\alpha\mathrm{d}p_\alpha=|\boldsymbol{\mathcal{J}}|\mathrm{d}\Omega \tag{13.6}$$

   (3) Jacobi 行列式：

$$|\boldsymbol{\mathcal{J}}|=\begin{vmatrix}\frac{\partial\widetilde{q}_\alpha}{\partial q_\beta} & \frac{\partial\widetilde{q}_\alpha}{\partial p_\beta}\\ \frac{\partial\widetilde{p}_\alpha}{\partial q_\beta} & \frac{\partial\widetilde{p}_\alpha}{\partial p_\beta}\end{vmatrix}=\begin{vmatrix}\delta_{\alpha\beta}+\frac{\partial^2\mathcal{H}}{\partial q_\beta\partial q_\alpha}\mathrm{d}t & \frac{\partial^2\mathcal{H}}{\partial p_\beta\partial p_\alpha}\mathrm{d}t\\ -\frac{\partial^2\mathcal{H}}{\partial q_\alpha\partial p_\beta}\mathrm{d}t & \delta_{\alpha\beta}-\frac{\partial^2\mathcal{H}}{\partial q_\alpha\partial p_\beta}\mathrm{d}t\end{vmatrix} \tag{13.7}$$

   (4) Jacobi 行列式展开：

$$|\boldsymbol{\mathcal{J}}|=1+\mathrm{tr}\begin{pmatrix}\frac{\partial^2\mathcal{H}}{\partial q_\beta\partial q_\alpha} & \frac{\partial^2\mathcal{H}}{\partial p_\beta\partial p_\alpha}\\ -\frac{\partial^2\mathcal{H}}{\partial q_\alpha\partial p_\beta} & -\frac{\partial^2\mathcal{H}}{\partial q_\alpha\partial p_\beta}\end{pmatrix}\mathrm{d}t+\mathcal{O}(\mathrm{d}t^2)=1+\mathcal{O}(\mathrm{d}t^2) \tag{13.8}$$

13.2 Possion 括号

1. Possion 括号 (Possion bracket)：

$$\{f,g\}=\sum_{\alpha=1}^{s}\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha}-\frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right)=\sum_{\alpha=1}^{s}\frac{\partial(f,g)}{\partial(q_\alpha,p_\alpha)} \tag{13.9}$$

2. Hamilton 正则方程：

$$\dot{q}_\alpha=\{q_\alpha,\mathcal{H}\},\quad \dot{p}_\alpha=\{p_\alpha,\mathcal{H}\} \tag{13.10}$$

3. 物理量的变化率：

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\sum_{\alpha=1}^{s}\left(\frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha\right)+\frac{\partial f}{\partial t}=\{f,\mathcal{H}\}+\frac{\partial f}{\partial t} \tag{13.11}$$

4. 重要的 Possion 括号：

$$\{q_\alpha,q_\beta\}=0,\quad \{p_\alpha,p_\beta\}=0,\quad \{q_\alpha,p_\beta\}=\delta_{\alpha\beta} \tag{13.12}$$

$$\{L_\alpha,x_\beta\}=\varepsilon_{\alpha\beta\gamma}x_\gamma,\quad \{L_\alpha,p_\beta\}=\varepsilon_{\alpha\beta\gamma}p_\gamma,\quad \{L_\alpha,L_\beta\}=\varepsilon_{\alpha\beta\gamma}L_\gamma \tag{13.13}$$

$$\{L_\alpha,\boldsymbol{x}^2\}=0,\quad \{L_\alpha,\boldsymbol{p}^2\}=0,\quad \{L_\alpha,\boldsymbol{L}^2\}=0 \tag{13.14}$$

14 正则变换

14.1 正则变换及其性质

1. 正则变换：相空间中的坐标变换 $\boldsymbol{x}(q,p)\to\boldsymbol{X}(Q,P)$ 满足 Hamilton 方程形式不变：

$$\dot{Q}_\alpha=\frac{\partial\mathcal{K}}{\partial P_\alpha},\quad \dot{P}_\alpha=-\frac{\partial\mathcal{K}}{\partial Q_\alpha} \tag{14.1}$$

2. 正则变换的性质：
   (1) 新旧 Possion 括号等价：

$$\{f,g\}_{(Q,P)}=\{f,g\}_{(q,p)} \tag{14.2}$$

   (2) Possion 括号结构不变：

$$\{Q_\alpha,Q_\beta\}=0,\quad \{P_\alpha,P_\beta\}=0,\quad \{Q_\alpha,P_\beta\}=\delta_{\alpha\beta} \tag{14.3}$$

14.2 正则变换的矩阵形式

1. 标准辛矩阵：

$$\boldsymbol{\Omega}=\begin{pmatrix}\boldsymbol{O} & \boldsymbol{I}_{s\times s}\\ -\boldsymbol{I}_{s\times s} & \boldsymbol{O}\end{pmatrix}=\boldsymbol{I}_{s\times s}\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\boldsymbol{I}_{s\times s}\otimes\mathrm{i}\boldsymbol{\sigma}_y \tag{14.4}$$

2. Hamilton 正则方程的矩阵形式：

$$\dot{\boldsymbol{x}}=\boldsymbol{\Omega}\frac{\partial\mathcal{H}}{\partial\boldsymbol{x}} \tag{14.5}$$

3. 对坐标变换求导：

$$\dot{\boldsymbol{X}}=\boldsymbol{\mathcal{J}}\dot{\boldsymbol{x}}=\boldsymbol{\mathcal{J}}\boldsymbol{\Omega}\frac{\partial\mathcal{H}}{\partial\boldsymbol{x}}=\boldsymbol{\mathcal{J}}\boldsymbol{\Omega}\boldsymbol{\mathcal{J}}^T\frac{\partial\mathcal{H}}{\partial\boldsymbol{X}} \tag{14.6}$$

4. 正则变换的 Jacobi 矩阵是辛矩阵：

$$\boldsymbol{\mathcal{J}}\boldsymbol{\Omega}\boldsymbol{\mathcal{J}}^T=\boldsymbol{\Omega} \tag{14.7}$$

14.3 正则变换的生成函数

1. 正则变换的充分条件：

$$\mathrm{d}F=\sum_{\alpha=1}^{s}(p_\alpha\mathrm{d}q_\alpha-P_\alpha\mathrm{d}Q_\alpha)+(\mathcal{K}-\mathcal{H})\mathrm{d}t \tag{14.8}$$

2. 第一类生成函数：

$$F_1(q,Q,t),\quad p_\alpha=\frac{\partial F_1}{\partial q_\alpha},\quad P_\alpha=-\frac{\partial F_1}{\partial Q_\alpha},\quad \mathcal{K}(Q,P,t)=\mathcal{H}(q,p,t)+\frac{\partial F_1}{\partial t} \tag{14.9}$$

3. 第二类生成函数：

$$F_2(q,P,t)=F_1+\sum_{\alpha=1}^{s}P_\alpha Q_\alpha,\quad p_\alpha=\frac{\partial F_2}{\partial q_\alpha},\quad Q_\alpha=\frac{\partial F_2}{\partial P_\alpha},\quad \mathcal{K}(Q,P,t)=\mathcal{H}(q,p,t)+\frac{\partial F_2}{\partial t} \tag{14.10}$$

4. 第三类生成函数：

$$F_3(p,Q,t)=F_1-\sum_{\alpha=1}^{s}q_\alpha p_\alpha,\quad q_\alpha=-\frac{\partial F_3}{\partial p_\alpha},\quad P_\alpha=-\frac{\partial F_3}{\partial Q_\alpha},\quad \mathcal{K}(Q,P,t)=\mathcal{H}(q,p,t)+\frac{\partial F_3}{\partial t} \tag{14.11}$$

5. 第四类生成函数：

$$F_4(p,P,t)=F_3+\sum_{\alpha=1}^{s}P_\alpha Q_\alpha,\quad q_\alpha=-\frac{\partial F_4}{\partial p_\alpha},\quad Q_\alpha=\frac{\partial F_4}{\partial P_\alpha},\quad \mathcal{K}(Q,P,t)=\mathcal{H}(q,p,t)+\frac{\partial F_4}{\partial t} \tag{14.12}$$

15 Hamilton-Jacobi 理论

15.1 Hamilton-Jacobi 方程

1. Hamilton 主函数：正则变换后的 Hamilton 量是 0，坐标是常数

$$Q_\alpha = \beta_\alpha, \quad P_\alpha = \alpha_\alpha, \quad S(q,t) = F_2(q,t) \tag{15.1}$$

2. Hamilton 主函数的正则变换：

$$p_\alpha = \frac{\partial S}{\partial q_\alpha}, \quad \beta_\alpha = \frac{\partial S}{\partial \alpha_\alpha}, \quad \mathcal{K}(Q,P,t) = \mathcal{H}(q,p,t) + \frac{\partial S}{\partial t} = 0 \tag{15.2}$$

3. Hamilton-Jacobi 方程：

$$\frac{\partial S}{\partial t} + \mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \tag{15.3}$$

15.2 Hamilton 特征函数的 Hamilton-Jacobi 方程

1. Hamilton 特征函数 $W(q)$：Hamilton 量不含时

$$S(q,t) = W(q) + V(t) = W(q) - \alpha_1 t \tag{15.4}$$

2. Hamilton 特征函数的正则变换：

$$p_\alpha = \frac{\partial W}{\partial q_\alpha}, \quad Q_\alpha = \frac{\partial W}{\partial \alpha_\alpha}, \quad \mathcal{K}(Q,P,t) = \mathcal{H}(q,p,t) = \alpha_1 \tag{15.5}$$

3. Hamilton 特征函数的 Hamilton-Jacobi 方程：

$$\mathcal{H}\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1 \tag{15.6}$$

4. Hamilton 正则方程的解：

$$P_\alpha = \alpha_\alpha, \quad Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1}, \quad Q_\alpha = \beta_\alpha = \frac{\partial W}{\partial \alpha_\alpha} \tag{15.7}$$

15.3 浸渐不变量

1. 一维系统的作用量-角变量：

$$I = \frac{1}{2\pi}\oint p\mathrm{d}q, \quad \dot{\theta} = \frac{\partial \mathcal{K}}{\partial I} = \omega \tag{15.8}$$

2. 浸渐不变量：

$$I = \frac{1}{2\pi}\oint p\mathrm{d}q \tag{15.9}$$

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